

Degrees of Rigidity for Souslin Trees*

Gunter Fuchs

Westfälische Wilhelms-Universität Münster, Germany

Joel David Hamkins[†]

The City University of New York, USA

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Abstract

We investigate various strong notions of rigidity for Souslin trees, separating them under \diamond into a hierarchy. Applying our methods to the automorphism tower problem in group theory, we show under \diamond that there is a group whose automorphism tower is highly malleable by forcing.

1 Introduction

Automorphisms and isomorphisms of ω_1 -trees have been long studied in set theory (see [DJ74], [Jec72], [Jec74], [Abr79], [AS85], [GS64]), and several of these authors have investigated various strong forms of rigidity of such trees. Here, by considering the absoluteness of rigidity and strong rigidity properties of a tree to various forcing extensions, we introduce several new rigidity concepts—all of which are exhibited by the generic Souslin trees added by the usual forcing—and separate them under \diamond into a proper implication hierarchy.

Our motivation for looking at these particular rigidity properties arose in connection with the automorphism tower problem in group theory. Specifically, the main result of [HT00] had made essential use of the absolute rigidity properties of generic Souslin trees to construct in the corresponding Souslin tree forcing extension a group whose automorphism tower is highly malleable by forcing. In the final section of this paper, we replace the Souslin tree forcing argument of [HT00] with a construction from \diamond , and we conclude, consequently, that there are such groups in the constructible universe L .

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Before introducing the rigidity notions in which we are interested, let us first set some background terminology, which we hope most readers will find familiar. Specifically, a *tree* is a partial order $T = \langle T, <_T \rangle$ in which the predecessors of any node are well ordered and there is a unique minimal element called the *root*. We will usually conflate the tree with its underlying set. The *height* of a node p in T , denoted $|p|$ or $|p|_T$, is the order type of its predecessors. We write $T(\alpha)$ for the α^{th} level of T , the set of nodes having height α . The *height* of a tree T , $\text{ht}(T)$, is the supremum of the heights of its nodes. We write $T|\alpha$ for the subtree of T of nodes having height less than α , and more generally, for any set S of ordinals, $T|S$ is the suborder of T consisting of the nodes on a level in S . For $\alpha \leq \omega_1$, an α -tree is a tree of height α with all levels countable. Such a tree is *normal* if every node has (at least) two immediate successors (except those on the top level, if α is a successor ordinal), nodes on limit levels are uniquely determined by their sets of predecessors, and every node has successors on all higher levels up to α . A tree is γ -*splitting* if every node has exactly γ many immediate successors. It is *uniform* if it is γ -splitting, for some γ . We write T_p to denote the subtree of T consisting of the nodes $q \in T$ with $q \geq_T p$. We write $T_{[p]}$ for the subtree of T consisting of the nodes $q \in T$ that are comparable with p . A branch in T is a maximal linearly ordered subset of T , and the *length* of the branch is its order type. We write $[T]$ for the set of cofinal branches, those branches containing nodes on every level. A tree T is *Aronszajn* if it is a normal ω_1 -tree with no cofinal branch. An *antichain* in a tree is a set of pairwise incomparable elements. A *Souslin* tree is a normal ω_1 -tree with no uncountable antichain. When forcing with a tree, we reverse the order, so that stronger conditions are higher up in the tree. Consequently, Souslin trees are c.c.c. as notions of forcing. It is well known that they are also countably distributive (see [Jec03a, Lemma 15.28]). An *automorphism* of a tree is an isomorphism of the tree with itself; it is *nontrivial* if it is not the identity function. We are now ready to define the various rigidity notions.

Definition 1.1 Suppose that T is a tree.

1. T is *rigid* if there is no nontrivial automorphism of T .
2. T is *totally rigid* if whenever p and q are distinct nodes in T , then T_p and T_q are not isomorphic.
3. T has the *unique branch property* (UBP) if $1 \Vdash_T T$ has exactly one new cofinal branch.
4. T is *absolutely rigid* if $1 \Vdash_T T$ is rigid.
5. T is *absolutely totally rigid* if $1 \Vdash_T T$ is totally rigid.
6. T is *absolutely UBP* if $1 \Vdash_T T$ has the UBP. Equivalently, forcing with $T \times T$ adds precisely 2 new cofinal branches.¹

¹See the proof of theorem 1.2.

7. For any property P , we say that T is *absolutely P* if $1 \Vdash_T T$ has property P , and more generally, T is \mathbb{Q} -*absolutely P* if $1 \Vdash_{\mathbb{Q}} T$ has property P .

We shall be interested primarily in the rigidity properties of uniform normal trees, because it is (too) easy to construct rigid non-uniform trees, simply by insisting that nodes on a level have distinct numbers of successors. But a simple back-and-forth argument shows that no uniform normal tree of countable height can be rigid – see [HT00, Lemma 3.6.]; the argument dates back to [Kur35]. Abraham [Abr79] showed (in ZFC) that a rigid Aronszajn tree exists. Note that forcing with an absolutely rigid tree must preserve ω_1 , since otherwise the tree would have countable height in the extension and hence fail to be rigid there. So the existence of an absolutely rigid Aronszajn tree cannot be proved from ZFC alone, since it is consistent that every Aronszajn tree is special and thus collapses ω_1 . Similarly, forcing with a normal tree with the unique branch property also preserves ω_1 , because every countable normal tree has continuum many branches. We will therefore concentrate, in our \diamond constructions, on building Souslin trees, the canonical candidates for trees preserving ω_1 .

We now observe some elementary implications between these rigidity notions.

Theorem 1.2 *Suppose that T is a normal tree.*

1. *If T is absolutely rigid or totally rigid, then it is rigid.*
2. *If T is absolutely totally rigid then it is totally rigid and absolutely rigid.*
3. *If T is UBP, then it is totally rigid.*
4. *If T is absolutely UBP, then it is UBP and absolutely totally rigid.*

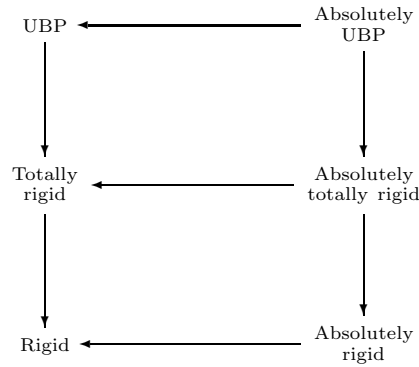


Figure 1: Implication Diagram

Proof. If T is absolutely rigid, then it is rigid, because a nontrivial automorphism of T in the ground model would still be a nontrivial automorphism in the

extension. If T is totally rigid, then it is rigid, because a nontrivial automorphism π of T that moves, say, p to q , would yield $\pi \restriction T_p : T_p \cong T_q$, contradicting total rigidity. So 1 holds. Similar reasoning shows that if T is absolutely totally rigid, then it is totally rigid, and also absolutely rigid, so 2 holds.

Suppose next that T has the unique branch property. If T were not totally rigid, then there would be distinct nodes p and q with an isomorphism $\pi : T_p \cong T_q$. We may assume without loss of generality that $|p| \leq |q|$. Thus, there is some extension $p' \geq p$ such that $p' \perp q$. Let b be V -generic for T with $p' \in b$. Thus, b provides a branch through T_p , and consequently $\pi[b]$ is a branch through T_q . Since $p' \perp q$, these branches are distinct, and so in $V[b]$, there are at least two branches through T , contradicting the unique branch property. So 3 holds.

Before continuing, let us first explain why the two definitions we gave of the absolute unique branch property are equivalent. If forcing with $T \times T$ necessarily adds only two new cofinal branches, then forcing with T and then forcing with T again clearly adds exactly one new branch each time, and so $1 \Vdash_T (1 \Vdash_T T \text{ has exactly one new cofinal branch})$. Conversely, it is enough to show that T has the unique branch property. But if this were not the case, then would be a name τ and a condition p forcing that it is a new cofinal branch, different from the generic branch.² In particular, one can strengthen p so as to decide τ in various incompatible ways. If $V[b_1][b_2]$ is obtained by forcing with $T \times T$ and $p \in b_2$, then it follows by a simple density argument that τ_{b_2} will be a cofinal branch not in $V[b_1]$, and different from b_2 , contradicting $1 \Vdash_T (1 \Vdash_T T \text{ has exactly one new cofinal branch})$. So the two definitions are in fact equivalent.

To prove 4, suppose T has the absolute unique branch property. Thus, forcing with $T \times T$ adds precisely two new branches b_1 and b_2 through T . If $V[b_1]$ already had 2 new cofinal branches through T , then there would be at least three such branches in $V[b_1, b_2]$, contrary to our assumption, and so T has the unique branch property in V . Lastly, we argue that T is absolutely totally rigid. If not, then in some extension $V[b]$ obtained by forcing with T , there would be an isomorphism $T_p \cong T_q$. As above, we may assume $p \perp q$. Further forcing to add a $V[b]$ -generic branch c containing p will also add an isomorphic copy of the branch through q , resulting that $V[b, c]$ has at least three new cofinal branches through T , contradicting our assumption that T was absolutely UBP. \square

Our main result is that no other implications are provable in ZFC.

Main Theorem 1.1 *The implication diagram of Figure 1 is complete. Namely, if ZFC is consistent, then no implication relations other than those appearing in the transitive closure of Figure 1 are provable in ZFC. Indeed, if \diamond holds, then there are Souslin trees exhibiting each of the non-implications of Figure 1.*

This theorem will be proved in Section 3. In Section 4, we place the diagram in a larger context including many other rigidity notions.

²When referring to *the* generic branch of a tree, we mean the canonical generic branch, the branch derived from the generic filter. So here, $p \Vdash \tau \neq \Gamma$, where Γ is the canonical name for the generic object. When T is Souslin, of course, *every* cofinal branch of T is generic, because every antichain is bounded by a level of the tree. See [DJ74, Thm. II.4.7].

2 Very rigid Souslin trees exist generically and under \diamond

Before proving that the implication diagram is complete, let us briefly show that the rigidity notions we have introduced actually occur. We will show that the generic Souslin trees one adds by the usual forcing exhibit all of the rigidity notions that we have mentioned above and more. And we will also construct such highly rigid trees under the hypothesis of \diamond .

The usual forcing to add a Souslin tree is the partial order \mathbb{P} consisting of all normal α -trees, subtrees of $2^{<\alpha}$, for any countable ordinal α , ordered by end-extension. If $G \subset \mathbb{P}$ is a V -generic filter, then the resulting tree $T = \bigcup G$ is called a *V-generic Souslin tree*. This forcing is countably closed. Souslin trees were first added by forcing in 1964 by Tennenbaum (see [Ten68]) and by Jech [Jec67].

Theorem 2.1 *Every V-generic Souslin tree T is in fact a Souslin tree in $V[T]$, and exhibits in $V[T]$ all of the rigidity properties appearing in Figure 1.*

Proof. Most of this proof is very well known (see [Jec03b, Thm. 15.23], [DJ74]); we include it for completeness and because it will motivate some of our later constructions. Suppose that $T = \bigcup G$ for a V -generic filter $G \subseteq \mathbb{P}$. It is easy to see that T is in fact an ω_1 -tree. Suppose that $A \subseteq T$ is a maximal antichain of T in $V[G]$. Let \dot{A} be a name for A , such that some condition $t_0 \Vdash \dot{A}$ is a maximal antichain in the generic tree T . For any condition t in \mathbb{P} , we may use the fact that \mathbb{P} is countably closed to find a stronger condition t' that decides $\dot{A} \cap t$. Further, we can find a stronger t'' such that for every node $a \in t$ there is a node $b \in t''$ comparable to it such that $t'' \Vdash b \in \dot{A}$. Iterating this, in what we call the *bootstrap* argument, we build a descending sequence $t_0 > t_1 > \dots$ in \mathbb{P} such that t_{n+1} decides $\dot{A} \cap t_n$ and t_{n+1} forces that every element of t_n is comparable to a node in $\dot{A} \cap t_{n+1}$. It follows that the limit tree $t_\omega = \bigcup t_n$ decides $\dot{A} \cap t_\omega$ and forces that it is a maximal antichain in t_ω . For each node $a \in t$, let b_a be a branch through t_ω containing a and passing through an element of the set $\dot{A} \cap t_\omega$ decided by t_ω . Let \bar{t} be $t \cup \{b_a \mid a \in t_\omega\}$ be the resulting tree. This is a condition in \mathbb{P} , stronger than t_ω , but any node on the top level of \bar{t} is comparable to something in $\dot{A} \cap t_\omega$. Thus, \bar{t} forces that no additional nodes can be added to the antichain \dot{A} above the height of \bar{t} : the antichain is “sealed”. Thus, \bar{t} forces that \dot{A} is countable, and so T is Souslin.

Next, we show that T has the unique branch property. We have to show that forcing with T adds a unique branch through T . The combined forcing that we are considering is $\mathbb{P} * \dot{T}$, where we first add the tree and then force with it. Given any condition $\langle t, \dot{q} \rangle$, we may strengthen t to a tree of successor height deciding the particular value of \dot{q} , and then strengthen that value to a node on the maximal level of t . Thus, this two-step forcing has a dense set D of conditions of the form $\langle t, \dot{q} \rangle$, where t is a normal $(\alpha + 1)$ -tree for some countable ordinal α and q is a node on the α^{th} level of t . The point is that forcing with $\mathbb{P} * \dot{T}$ is equivalent to forcing with D , since D is dense, but D has the advantage

of being countably closed. Suppose that τ is a D -name for another cofinal branch through T , and this is forced by some condition $\langle t_0, p_0 \rangle \in D$. Extend this to a stronger condition $\langle t_1, p_1 \rangle \in D$ that decides $\tau \cap t_0$, and so on to build $\langle t_n, p_n \rangle > \langle t_{n+1}, p_{n+1} \rangle$, with the next condition deciding τ on the previous tree. Let $t_\omega = \bigcup t_n$ be the limit tree and $p_\omega = \bigcup p_n$ be the limit of the nodes p_n in this tree. Let $b = \tau_{\langle t_\omega, p_\omega \rangle}$ be the branch through t_ω decided by the conditions $\langle t_n, p_n \rangle$. Extend t_ω to a tree \bar{t} by adding branches through every node, to form a maximal level, but without adding the branch b . Thus, $\langle \bar{t}, p_\omega \rangle$ is a condition in D , stronger than $\langle t_0, p_0 \rangle$, forcing that τ cannot continue past this level, contrary to our assumption that τ was forced to name a new cofinal branch.

The proof that T has the absolute unique branch property is similar. Now, we are forcing with $\mathbb{P} * \dot{T} * \dot{T}$, and there is a dense set D of conditions of the form $\langle t, p_1, p_2 \rangle$, where t is a normal $(\alpha + 1)$ -tree for some countable α and p_1 and p_2 are two nodes in $t(\alpha)$. If τ named a new cofinal branch, one could carry out the previous argument to build a condition deciding this branch, but then extend the tree so as to prevent this new branch from extending. \square

A similar argument, going beyond the notions considered in Figure 1, shows that a generic Souslin tree T is *n-absolutely UBP* for every natural number n , meaning that forcing with T^{n+1} adds exactly $n+1$ new cofinal branches through T . In fact, generic Souslin trees exhibit even stronger rigidity properties:

Definition 2.2 A Souslin tree T is *Souslin off the generic branch* if after forcing with T to add a generic branch b , then T_p remains Souslin for every node p not on b . More generally, T has property P *off the generic branch* if after forcing with T to add a generic branch b , then every T_p has property P for $p \notin b$. We say that T is *n-fold Souslin off the generic branch* if after forcing with T^n , which adds n branches b_1, \dots, b_n , then T_p remains Souslin for any p not on any b_i .

Thus, another way to say that T has the unique branch property is to say that it is Aronszajn off the generic branch. In particular, if a tree is Souslin off the generic branch, then it will have the unique branch property, since a second branch would betray the Souslin-ness of that part of the tree. Analogously, if a tree is n -fold Souslin off the generic branch, it must be n -absolutely UBP.

The other notions of rigidity that we introduced can be strengthened analogously. Given a property P , we say that T is *n-absolutely P* if T has property P in every generic extension obtained by forcing with T^n . Then the remarks about the preservation of ω_1 carry over. Thus, if a tree T is n -absolutely rigid, then forcing with T^n preserves ω_1 , and if T is n -absolutely UBP, then forcing with T^{n+1} preserves ω_1 .

Theorem 2.3 *Every V-generic Souslin tree is Souslin off the generic branch. Indeed, such trees are n-fold Souslin off the generic branch for every natural number n. Consequently, they are also n-absolutely UBP for every $n < \omega$, and n-absolutely rigid, and so on.*

Proof. Suppose that we force with \mathbb{P} to add a V-generic Souslin tree T and then force with the tree T itself. This two-step forcing is equivalent to the

forcing $\mathbb{P} * \dot{T}$, where \dot{T} is the \mathbb{P} -name for the resulting generic tree added by \mathbb{P} . We have seen already that this forcing has a dense set D consisting of normal $(\alpha+1)$ -trees t and nodes b on the α^{th} level $t(\alpha)$. This is countably closed forcing. Suppose $\langle t, q \rangle$ is such a condition in D and p is a node in t incomparable to q . We claim that the resulting tree T_p is a Souslin tree in the extension $V[T, b]$. To see this, suppose that \dot{A} is a D -name for an antichain in T_p . The bootstrap argument of Theorem 2.1 shows that any condition in D can be extended to a stronger condition forcing that \dot{A} is bounded. Hence, T_p is Souslin in $V[T, b]$, and so T is Souslin off the generic branch.

The n -fold version is similar. Forcing with \mathbb{P} to add a Souslin tree T and then forcing with T^n is the same as forcing with $\mathbb{P} * (\dot{T})^n$. This poset has a dense set D that is isomorphic to the set of conditions of the form $\langle t, q_1, \dots, q_n \rangle$, where t is a normal $(\alpha+1)$ -tree for some countable ordinal α and q_1, \dots, q_n are n many nodes on the α^{th} level $t(\alpha)$. This forcing is countably closed. If p is a node in t and incomparable to each q_i , then the argument of the previous paragraph shows that the resulting subtree T_p will still be Souslin in $V[T, b_1, \dots, b_n]$, as desired. \square

Let us now turn to the construction of Souslin trees from \diamond . This is the combinatorial principle asserting that there is a sequence $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ such that for every $A \subseteq \omega_1$ the set $\{\alpha \mid A \cap \alpha = D_\alpha\}$ is stationary.

Theorem 2.4 (Jensen) *If \diamond holds, then there is a rigid Souslin tree.*

Proof. This construction is widely known (see [Jec03a] or [DJ74]), but we include it as a warm-up to Theorem 2.5 and the arguments of Sections 3 and 5. Fix any \diamond sequence $\langle D_\alpha \mid \alpha < \omega_1 \rangle$, using countable ordinals for the nodes. We first explain merely how to construct a Souslin tree T , by recursively constructing the levels $T(\alpha)$. We begin with a sole root node. At successor stages, if the current top level $T(\alpha)$ is defined, then we give each of these nodes two immediate successors in $T(\alpha+1)$. The nontrivial case occurs when $T|\alpha$ is defined and α is a limit ordinal. If D_α happens to be a maximal antichain in $T|\alpha$, then for every node $p \in T|\alpha$, select a branch $b_p \in [T|\alpha]$ with $p \in b_p$ and $b_p \cap D_\alpha \neq \emptyset$. For each such branch b_p , place a node in $T(\alpha)$ on top of it. In this case, we say that $T(\alpha)$ has *sealed* the antichain D_α . If D_α is not a maximal antichain in $T|\alpha$, then carry out the same construction, but dropping the requirement involving D_α . The resulting tree T is clearly a normal ω_1 -tree. To see that it is Souslin, suppose that $A \subset T$ is a maximal antichain. By some simple closure arguments, there is a club of α such that $A \cap \alpha$ is a maximal antichain in $T|\alpha$. By \diamond , the set of α with $A \cap \alpha = D_\alpha$ is stationary, and so there is an $\alpha < \omega_1$ such that $D_\alpha = A \cap \alpha$ is a maximal antichain in $T|\alpha$. In this case, we sealed the antichain, and every node in $T(\alpha)$ lies above an element of $A \cap (T|\alpha)$. Therefore, all elements of T above level α are comparable to an element of $A \cap (T|\alpha)$, and so $A = A \cap (T|\alpha)$ is countable, as desired.

To ensure that T is rigid, we now fold an additional step into the construction. At a limit stage α , if D_α happens to code a nontrivial automorphism f of the tree $T|\alpha$, then we find a branch $b \in [T|\alpha]$ such that $f[b] \neq b$. For each

$p \in T|\alpha$, select a branch $b_p \in [T_p|\alpha]$ with $b_p \neq f[b]$, and let $T(\alpha) = \{b_p \mid p \in T|\alpha\} \cup \{b\}$. Note that because we have included b but not $f[b]$ in $T(\alpha)$, we have prevented f from being the initial segment of an automorphism of $T|(\alpha+1)$. It follows that the final tree T is rigid, because if $f : T \cong T$ were a nontrivial automorphism of the full tree, then there would be a stationary set of α such that D_α codes $f \restriction T|\alpha$, and at such a stage, once we are into the nontrivial part of f , we specifically constructed the α^{th} level so as to exclude the possibility that $f \restriction (T|\alpha)$ could be extended to an automorphism of $T|(\alpha+1)$. Since this contradicts our assumption that f was an automorphism of T , we have constructed a rigid Souslin tree. \square

It was also shown by Jensen that \diamond implies the existence of Souslin trees that fail to be rigid, in a very strong way, by satisfying strong forms of homogeneity (see [DJ74]).

The key method in the proof of Theorem 2.4 was to anticipate via \diamond the potential antichains and automorphisms of the tree, and seal them into a level of the tree, preventing them from growing into full counterexamples. An elaboration of this method allows us also to attain the higher degrees of rigidity.

Theorem 2.5 *If \diamond holds, then there is a Souslin tree that is n -fold Souslin off the generic branch, for every finite n , and consequently also n -absolutely UBP and n -absolutely rigid and so on.*

Proof. We again construct the tree T by induction on the levels $T|\alpha$, using a fixed \diamond sequence $\langle D_\alpha \mid \alpha < \omega_1 \rangle$. We begin as before with a sole root node, and at successors we give every node two immediate successors. So assume that α is a limit ordinal and let $t = T|\alpha$ be the tree constructed up to stage α . While we could anticipate and then seal potential antichains and potential automorphisms, it will be unnecessary to do so explicitly. Rather, it will suffice to anticipate and then seal the potential violations to the tree being n -fold Souslin off the generic branch. Specifically, suppose that for some natural number n and some $\vec{q} \in t^n$, the set D_α codes a $t_{[\vec{q}]}$ -name \dot{A} for a maximal antichain in t_p for some $p \perp q_i$. More precisely, associated to this name is the function $\vec{r} \mapsto \dot{A}_{\vec{r}}$, where $\dot{A}_{\vec{r}}$ is the set of elements in t_p that the condition \vec{r} determines to be in the name \dot{A} . We assume that every $\dot{A}_{\vec{r}}$ is an antichain in t_p ; if \vec{s} extends \vec{r} , then $\dot{A}_{\vec{r}} \subseteq \dot{A}_{\vec{s}}$; and for any $q \in t_p$ and any \vec{r} extending \vec{q} , there is some \vec{s} extending \vec{r} such that $\dot{A}_{\vec{s}}$ contains an element compatible with q in t_p . Using these properties and the fact that t_p is countable, we may successively extend \vec{q} to meet a certain countable collection of dense sets so as to build a cofinal branch $\vec{b} \in [t^n]$ extending \vec{q} so that the corresponding combined information $\dot{A}_{\vec{b}} = \bigcup \{\dot{A}_{\vec{b} \restriction \beta} \mid \beta < \alpha\}$ determined by the nodes on this branch is a maximal antichain in t_p . Since $p \perp q_i$, the node p is not on any of the branches b_i . Now, for every node q in $t_{[p]}$, we find a branch $b_q \in [t_{[q]}]$ going through the maximal antichain $\dot{A}_{\vec{b}}$. For each node $q \in t$ incomparable with p , we find any branch $b_q \in [t_{[q]}]$. Let B be the resulting combined set of branches, including \vec{b} and all

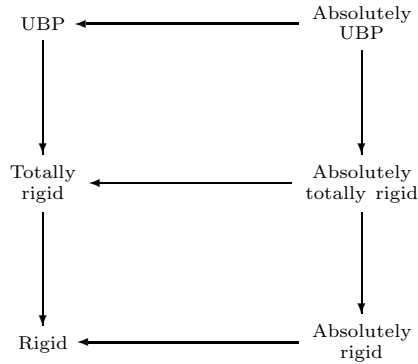
b_q for $q \in t$, and define $T(\alpha)$ to have nodes above exactly the branches in B . By construction, this is a normal $(\alpha + 1)$ -tree.

We now argue for every finite n that the resulting tree T is n -fold Souslin off the generic branch. Suppose for some n , some condition $\vec{q} = \langle q_1, \dots, q_n \rangle \in T^n$ and some $p \perp q_i$, there is a T^n -name \dot{A} such that \vec{q} forces that \dot{A} is a maximal antichain in T_p . The name \dot{A} is coded by a subset $\dot{A}_0 \subseteq \omega_1$. There is a club of α such that $T \cap \alpha = T|_\alpha$, and $\dot{A}_0 \cap \alpha$ is the code for $\dot{A} \cap (T|_\alpha)$, and such that this is actually a $(T|_\alpha)^n$ -name for a maximal antichain in $(T|_\alpha)_p$. By \diamond , therefore, there is some α like this such that in addition, D_α is the code of $\dot{A} \cap (T|_\alpha)$. Therefore, our construction above exactly ensures that $T(\alpha)$ has nodes $\vec{b} = \langle b_1, \dots, b_n \rangle$ that already determine $\dot{A} \cap (T_p|_\alpha)$ to be a maximal antichain in $T_p|_\alpha$, and furthermore, that every node in $T_p(\alpha)$ passed through a node in the set $\dot{A}_{\vec{b}} = \dot{A} \cap (T_p|_\alpha)$ determined by \vec{b} . Thus, \vec{b} forces via T^n that \dot{A} has been sealed by level α , and consequently that $\dot{A} = \dot{A}_{\vec{b}}$. So \vec{q} did not force that \dot{A} was unbounded. Consequently, since n , p and \vec{q} were arbitrary, we have established that T is n -fold Souslin off the generic branch for every n . It follows that T is Souslin, and rigid, and n -absolutely rigid, and n -absolutely UBP and so on. \square

We close this section by remarking that we have checked in detail that Jensen's original tree (see the construction of [DJ74, Thm. V.1.1]) is also n -fold Souslin off the generic branch.

3 Separating the rigidity notions

We turn now to our main result, proving that the diagram of implications in Figure 1 is complete and omits no ZFC-provable implication. For convenience, we reproduce the diagram here.



Observation 3.1 *In order to show that the implication diagram above is complete, it suffices to find*

1. *An absolutely rigid tree that is not totally rigid,*
2. *An absolutely totally rigid tree that is not UBP, and*
3. *A UBP tree that is not absolutely rigid.*

Proof. Assume there are trees exhibiting 1, 2 and 3. A tree as in 1 shows that there are no implications from the bottom row to the middle row. A tree as in 2 shows that there are no implications from the middle row to the top row. And a tree as in 3 shows that there are no implications from the left column to the right column. This refutes any potential implication either going (in any sense) up or to the right. It follows that there are no missing implications, because the transitive closure of the diagram already exhibits all implications going down and to the left. \square

So we will prove Theorem 1.1 by finding trees fulfilling the requirements of Observation 3.1.

Theorem 3.2 *If there is an absolutely rigid Souslin tree, then there is an absolutely rigid Souslin tree that is not totally rigid.*

This is a consequence of the following more general lemma.

Lemma 3.3 *If T is a tree, then there is a tree $C(T)$ such that:*

1. *T is rigid if and only if $C(T)$ is rigid.*
2. *T is absolutely rigid if and only if $C(T)$ is absolutely rigid.*
3. *T is Souslin if and only if $C(T)$ is Souslin.*
4. *$C(T)$ is not totally rigid.*
5. *T and $C(T)$ are forcing equivalent.*

Proof. The tree $C(T)$ is built by gluing ω copies of T together as in figure 2. Denoting the n^{th} copy of T by T_n , we add the new nodes r_1, r_2, \dots , having no limit in the tree, and use them to glue the copies of T together in such a way that r_0 is the root of the new tree, and the immediate successors of r_n are the root of T_n and r_{n+1} . Clearly, antichains in $C(T)$ consist essentially of copies of ω many antichains in T , and so 3 holds by the pigeonhole principle. The tree $C(T)$ is not totally rigid, since it is built from copies of T , and so 4 holds. Forcing with $C(T)$ amounts to selecting a copy of T and forcing with it, and so 5 holds.

Consider statement 1. Clearly any nontrivial automorphism of T gives rise to a nontrivial automorphism of $C(T)$, so T is rigid if $C(T)$ is. Conversely, suppose T is rigid and π is a non-trivial automorphism of $C(T)$. Since each T_n

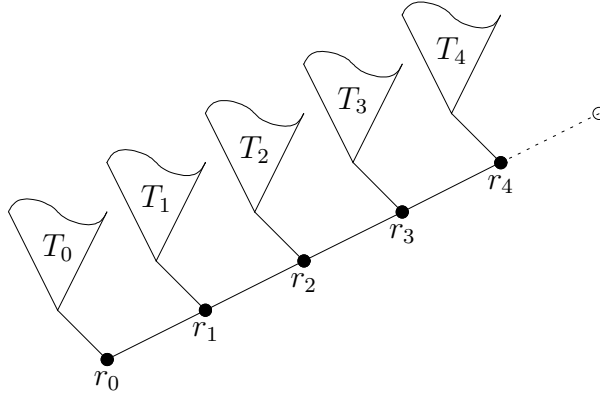


Figure 2: The tree $C(T)$ has ω many copies of T

is a copy of T and hence rigid, it follows easily that π must move the root of some T_n to the adjacent r_{n+1} . Thus, π witnesses that T_n is isomorphic to $C(T)_{r_{n+1}}$. By construction, however, $C(T)_{r_{n+1}}$ is isomorphic to $C(T)$. We conclude that T_n , and hence T , is isomorphic to $C(T)$. But this contradicts our assumptions that T was rigid, while $C(T)$ was not. So 1 holds.

Lastly, consider statement 2. By 5, forcing with T or $C(T)$ gives rise to the same extension. Applying 1 in such an extension, since the definition of $C(T)$ is absolute, we conclude that T is rigid there if and only if $C(T)$ is rigid. Thus, in the ground model, T was absolutely rigid if and only if $C(T)$ was absolutely rigid. So 2 holds. \square

This proves Theorem 3.2 and fulfills the first requirement of Observation 3.1. We turn now to the second requirement.

Theorem 3.4 *If there is a (Souslin) tree with the absolute unique branch property, then there is an absolutely totally rigid (Souslin) tree without the unique branch property.*

To prove this theorem, we introduce another notion of (non)rigidity: a structure is *Hopfian* if it is isomorphic to a proper substructure of itself. The following Lemma answers a question by Martin Weese (private communication).

Lemma 3.5 *Normal UBP trees are not Hopfian.*

Proof. Assume the contrary. Suppose T is a normal UBP tree, $A \subsetneq T$ and $\pi : T \cong T \upharpoonright A$. We claim that there is a $p \in T$ such that $\pi(p) \perp p$. Such a p forces via T that there are at least two new cofinal branches, for if b is a generic branch of T through p , then $\pi[b]$ determines a cofinal branch through $\pi(p)$, and as $p \perp \pi(p)$, these branches are different. This contradicts the unique branch property. To find such a p , let $\bar{p} \in T \setminus A$. If $\pi(\bar{p}) \perp \bar{p}$, we're done. Otherwise,

$\pi(\bar{p}) > \bar{p}$. By the normality of T , let $p \in T$ be such that $p > \bar{p}$ and $p \perp \pi(\bar{p})$. Then $\pi(p) > \pi(\bar{p})$, and so $p \perp \pi(p)$, because $p \perp \pi(\bar{p})$, as desired. \square

Lemma 3.6 *If T is a normal ω_1 -tree, then there is a normal ω_1 -tree $C'(T)$ such that:*

1. $C'(T)$ is not UBP.
2. If T is UBP, then $C'(T)$ is totally rigid.
3. If T is absolutely UBP, then $C'(T)$ is absolutely totally rigid.
4. T is Souslin if and only if $C'(T)$ is Souslin.
5. $C'(T)$ is ω -splitting.

Proof. Let T be a normal ω_1 -tree. We define a descending sequence $\langle C_n \mid n < \omega \rangle$ of club subsets of ω_1 as follows. For any subset A of ω_1 , let A' be the set of limit points of A below ω_1 . Let $C_0 = \{0\} \cup \omega'_1$, and recursively define $C_{n+1} = \{0\} \cup C'_n$. Let $C'(T)$ be the tree sketched in figure 3. More precisely, $C'(T)$ consists of

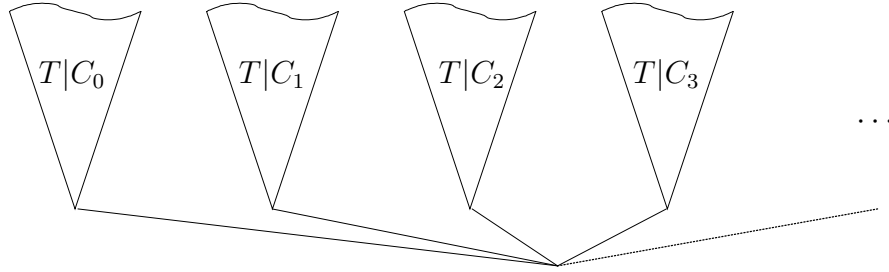


Figure 3: The tree $C'(T)$

nodes $\{0\} \cup \bigcup_{n \in \omega} (\{n\} \times (T|C_n))$. Note that $T|C_n$ is an ω -splitting normal ω_1 tree, since successive elements of C_n jump over gaps of length at least ω , and the limit nodes are unique since C_n is closed. It follows that $C'(T)$ is also a normal ω -splitting tree, so statement 5 holds. Since uncountable antichains in T give rise to uncountable antichains in $T|C$ and vice versa, statement 4 holds. Since every $T|C_n$ is dense in T , it follows that forcing with T is equivalent to forcing with $C'(T)$. Clearly, adding a branch to T will add ω many branches to $C'(T)$, so statement 1 holds. Statement 3 follows from statement 2, since forcing with T or $C'(T)$ gives rise to the same extensions, and so if T has the unique branch property in such an extension, then by statement 2 we know $C'(T)$ is also totally rigid there, as the definition of $C'(T)$ is absolute.

Lastly, consider statement 2, and suppose that T has the unique branch property. It follows easily that every $T|C_n$ also has the unique branch prop-

erty.³ By Lemma 1.2, therefore, every $T|C_n$ is also totally rigid. Suppose towards contradiction that $C'(T)$ is not totally rigid. So there is an isomorphism $\pi : C'(T)_{\langle i, \bar{p} \rangle} \cong C'(T)_{\langle j, \bar{q} \rangle}$ witnessing this. Since $T|C_i$ is totally rigid, it must be that $i \neq j$, and we may assume $i < j$. We claim that there is an extension $\langle i, p \rangle$ of $\langle i, \bar{p} \rangle$ whose image $\langle j, q \rangle = \pi(\langle i, p \rangle)$ has $|p|_T = |q|_T$. To see this, let $\bar{\gamma}$ and $\bar{\gamma}'$ be the continuous monotone enumerations of the clubs $C_i \setminus |\bar{p}|_T$ and $C_j \setminus |\bar{q}|_T$, respectively. The fixed points of these enumerations form a club, and whenever α is such a common fixed point and $|r|_T = \alpha$, then $|\pi(\langle i, r \rangle)|_T = \alpha$ as well. So any $p >_T \bar{p}$ at such a level α will have $|p|_T = |q|_T$, as we claimed. Since $\pi(\langle i, p \rangle) = \langle j, q \rangle$, the isomorphism π induces an isomorphism $\pi' : (T|C_i)_p \cong (T|C_j)_q$ on the underlying trees, defined by $\pi'(r) = r'$ if and only if $\pi(\langle i, r \rangle) = \langle j, r' \rangle$. If $p \neq q$, then $p \perp \pi'(p)$ and so forcing with the condition p in $T|C_n$ will add at least two new branches: the generic branch and the branch containing its image under π' . This contradicts the fact that $T|C_n$ is UBP. Otherwise, we assume $p = q$, and so π' is an isomorphism of $(T|C_i)_p$ with a proper subtree of itself. By Lemma 3.5, this also contradicts the unique branch property of $(T|C_i)_p$. So statement 2 holds, and the proof is complete. \square

This proves Theorem 3.4 and fulfills the second requirement of Observation 3.1. We turn now to the third and most difficult requirement. For this, we will assume \diamond and construct a Souslin tree with the unique branch property, but which is not absolutely rigid. The basic idea will be to construct a tree T such that cofinal branches through T automatically code automorphisms of T . Thus, forcing with T will necessarily add automorphisms to T . The difficulty will be to do this while retaining the unique branch property. To assist with our construction, we introduce the concept of level-transitive group actions on trees.

Definition 3.7 Let T be a tree and G be a group. Then a group action of G on T *respects* T for every $g \in G$, the function $p \mapsto g.p$ is an automorphism of T (which we henceforth denote by g). The action is *level-transitive* if for every $\alpha < \text{ht}(T)$, the induced group action on $T(\alpha)$ is transitive, meaning that for every two nodes $p, q \in T(\alpha)$, there is a $g \in G$ such that $g.p = q$.

An equivalent way of saying that the action is level transitive is that the orbit $G[p] = \{g.p \mid g \in G\}$ of any node p in the tree under the group action is simply the corresponding level $T(|p|)$ of the tree. It follows that for any cofinal branch through the tree, the images of this branch under the group action yield the entire tree. In the following, we will mainly be concerned with groups of automorphisms of a tree T and their canonical actions on T , namely, the action $\pi.p = \pi(p)$.

³In fact, if C is a club subset of ω_1 containing 0, then T has the unique branch property if and only if $T|C$ also has it. So, the unique branch property is a very natural notion: In [Jec74], it was shown that when investigating forcing extensions obtained by forcing with a Souslin tree T , the rigidity properties of $T|C$ matter, not those of T . I.e., automorphisms of the complete Boolean algebra associated to T correspond to automorphisms of $T|C$, for some C as above – see [Jec74, Lemma 3.1.].

Definition 3.8 A set of cofinal branches B through a tree T *covers* the tree if $\bigcup B = T$, so that every node in T lies on a branch in B . The set B *respects* an automorphism π of T if B is closed under pointwise application of π ; that is, if whenever $b \in B$, then $\pi[b] \in B$. The set B *respects* a group action of a group G on T , if it respects the associated automorphisms of every element of G under the group action. If b is a cofinal branch of T and $g \in G$, then let $g[b]$ be the image of b under the automorphism associated to g . We shall write $G[b]$ for the orbit of b under the group action, namely, $G[b] = \{g[b] \mid g \in G\}$. Thus, B respects G if and only if $G[b] \subseteq B$, for every $b \in B$.

When constructing our trees from \diamond , we will follow the same basic strategy as in Theorem 2.4, in that we will use the diamond sequence to anticipate antichains or automorphisms or names of automorphisms, and so on, and then extend the tree so as to kill off or seal these anticipated objects. By doing so, we will ensure that the ω_1 -tree we ultimately construct will have the desired rigidity properties. We explain in Definition 3.9 exactly the sense in which we will seal these various objects; one should imagine here that we propose to extend a countable tree T by adding a limit level containing nodes exactly above the branches in B .

Definition 3.9 Let T be a tree of limit height and B a set of branches that covers T .

1. B *seals a maximal antichain* A of T if for every $b \in B$, $b \cap A \neq \emptyset$.
2. B *seals a nontrivial automorphism* π of T if B does not respect it. This means that there is a branch $b \in B$ such that $\pi[b] \notin B$.
3. A function f is a *potential additional branch* of T if for some $p \in T$ it is an order preserving map $f : T_p \longrightarrow T$ with:
 - (a) $f(p) \perp p$.
 - (b) $\forall \gamma < \text{ht}(T) \forall q \geq_T p \exists r \geq_T q \quad |f(r)|_T \geq \gamma$.
 - (c) $\forall q \geq_T p \exists r_0, r_1 \geq_T q \quad f(r_0) \perp f(r_1)$.
4. B *seals a potential additional branch* if there is a $b \in B$ such that $f[b]$ determines a cofinal branch through T (the closure of $f[b]$ under $<_T$), but this branch is not in B .
5. A function f is a *potential additional automorphism* if there is a $p \in T$ such that $\text{dom}(f) = T_p$ and:
 - (a) For all $q \geq_T p$, $f(q)$ is a partial automorphism of T .
 - (b) f is monotonic, meaning that $p \leq_T q \leq_T r$ implies $f(q) \subseteq f(r)$.
 - (c) For all $q \geq_T p$, there are $r_0, r_1 \geq_T q$ such that there exists an $s \in \text{dom}(f(r_0)) \cap \text{dom}(f(r_1))$ with the property that $f(r_0)(s) \neq f(r_1)(s)$.
 - (d) For all $q \geq_T p$ and all $r \in T$, there is a $q' \geq_T q$ such that $r \in \text{dom}(f(q'))$.

6. B seals a potential additional automorphism f if there is a $b \in B$ such that B does not respect $\bigcup f[b]$. This means that there is a branch $c \in B$ such that $c \subseteq \text{dom}(\bigcup f[b])$ and $(\bigcup f[b])[c] \notin B$.
7. The notions *potential additional branch of degree n* and *potential additional automorphism of degree n* are defined analogously. Thus, a potential additional branch of degree n is an order preserving function $f : T_{p_0} \times \cdots \times T_{p_{n-1}} \longrightarrow T$ such that $f(\vec{p}) \perp p_i$ for all $i < n$ and:
 - (a) $\forall \gamma < \text{ht}(T) \forall \vec{q} \geq_{T^n} \vec{p} \exists \vec{r} \geq_{T^n} \vec{q} \quad |f(\vec{r})|_T \geq \gamma$.
 - (b) $\forall \vec{q} \geq_T \vec{p} \exists \vec{r}^0, \vec{r}^1 \geq_{T^n} \vec{q} \quad f(\vec{r}^0) \perp f(\vec{r}^1)$.

Lemma 3.10 *Let T be a normal tree of limit height.*

1. T is UBP if and only if there is no potential additional branch of T . Analogously, T is n -absolutely UBP if and only if there is no potential additional branch of degree n .
2. Forcing with T adjoins no new automorphisms of T if and only if there is no potential additional automorphism of T . Again, forcing with T^n adds no new automorphism of T if and only if there is no potential additional automorphism of degree n .

Proof. Statement 1 is proved by realizing that a potential additional branch of T is essentially the same as a T -name for a new branch different from the generic branch. More precisely, suppose T does not have the unique branch property, so that there is a condition $p \in T$ forcing that τ is (the name of) a new cofinal branch through T , different from the generic branch. Since p forces that τ is different from the generic branch, we may strengthen p if necessary and assume that $p \Vdash r \in \tau$ for some $p \perp r$. More generally, for each $q \in T_p$, let $f(q)$ be the T -maximal node r such that $q \Vdash_T r \in \tau$ (such a maximal r exists by the uniqueness of limit nodes in T). It is now easy to see that this is a potential additional branch. By design, $f : T_p \rightarrow T$ is order preserving and $f(p) \perp p$. For any $\gamma < \text{ht}(T)$ and any $q \in T_p$, we may extend q to some q' so as to decide τ beyond height γ , so $\gamma \leq |f(q')|$. And since τ is forced to be not in \check{V} , there is no condition deciding all of it; so for every $q \in T_p$ there are extensions q_0 and q_1 forcing specific incompatible nodes into τ , so that $f(q_0) \perp f(q_1)$.

Conversely, suppose that $f : T_p \longrightarrow T$ is a potential additional branch. We claim that p forces that at least two new branches are added. To see this, suppose that b is a V -generic branch through T containing p . Let c be the closure of $f[b]$. Some simple density arguments show that c is a cofinal branch through T that is not in V . And since $p \perp f(p)$, we know $b \neq c$. Thus, T does not have the unique branch property, establishing the first claim of 1. The statement for arbitrary finite n follows in the same way.

A similar argument establishes statement 2. Specifically, if some $p \in T$ forces that π is (the name of) a new automorphism of T , then for each $q \in T_p$ we define the corresponding partial isomorphism $f(q)$ of T by:

$$f(q)(s) = t \quad \text{if and only if} \quad q \Vdash \pi(\check{s}) = \check{t}.$$

It is easy to see that f is a potential additional automorphism of T . Conversely, if we have a potential additional automorphism f , on domain T_p , then we force below p to add a V -generic branch b . Some simple density arguments now establish that $\pi = \bigcup_{q \in b, q \geq p} f(q)$ is a new automorphism of T . The argument for general finite n is similar. \square

Now we are ready to state our general sealing lemma.

Lemma 3.11 *Let T be a countable normal tree of limit height. Let G be a countable group respecting T and acting level-transitively on T .*

1. *If b is a cofinal branch of T , then $G[b]$ is a countable set of branches covering T and respecting G .*
2. *Every maximal antichain in T can be sealed by a countable set of branches covering T and respecting G . Moreover, this set can be chosen to be the orbit of a single cofinal branch under G .*
3. *Let π be a nontrivial automorphism of T . Then π can be sealed by a countable set of branches covering T .*

If there is a $p_0 \in T$ such that $\pi(p_0) \neq p_0$ and G satisfies the requirement

$$(\Gamma) \quad \forall \sigma \in G \quad \forall p \geq_T p_0 \quad \exists p' \geq_T p \quad \sigma(p') \perp \pi(p'),$$

then there is a countable set of branches sealing π , covering T and respecting G . Moreover, this set can be chosen to be the orbit of a single branch under the group action.

4. *Every potential additional branch f can be sealed by a countable set of branches covering T .*

If G satisfies condition (Γ) at some $p_0 \in T$, with f replacing π , then there is a countable set of branches sealing f , covering T and respecting G . Again, this set can be chosen to be the orbit of a single branch.

5. *Every potential additional branch f of degree $n \in \omega$ can be sealed by a countable set of branches covering T .*
6. *For any countable set B of branches through T , there is a countable set of branches B' , disjoint from B , which covers T , respects G , and is the orbit of a single branch.*

Proof. 1.) Let b be a cofinal branch of T , and let $B = G[b]$ be the corresponding orbit of b under the action of G . Clearly, B is a countable set of cofinal branches respecting G . To see that B covers T , suppose $s \in T$ and let t be the $|s|_T^{\text{th}}$ element of b . Since the group action is level transitive, there is $\sigma \in G$ such that $\sigma(t) = s$, and consequently, $s \in \bigcup B$.

2.) Let A be a maximal antichain. For each $\sigma \in G$, let:

$$D_\sigma = \{q \in T \mid \exists a \in A \quad \sigma(q) \geq_T a\}.$$

Each D_σ is dense in T , viewed as a notion of forcing. To see this, for any node p use the maximality of A to find a $q' \geq_T \sigma(p)$ such that $q' \geq_T a$ for some $a \in A$, and then observe that $q \stackrel{\text{def}}{=} \sigma^{-1}(q')$ is in D_σ and $q \geq_T p$, as desired. Using this, we may now construct a cofinal branch b through T meeting each of the countably many dense sets D_σ for $\sigma \in G$. Let $B = G[b]$. By 1, B is a countable cover of T that respects G . We argue that B seals A . For any $\sigma \in G$, since the branch b contains some $p \in D_\sigma$, we know $\sigma(p)$ is above an element of A , and so every $\sigma[b] \in B$ intersects A , as desired.

3.) Suppose that π is a nontrivial automorphism of T . Fix any branch b such that $b \neq \pi[b]$, and for each $p \in T$ choose a branch b_p such that $p \in b_p \neq \pi[b]$, which is possible because T is ever-branching. The set $B = \{b_p \mid p \in T\} \cup \{b\}$, therefore, is a countable set of branches sealing π and covering T , as desired. For the more general claim, suppose that a group G acts level transitively on T and satisfies (Γ) with respect to p_0 . For $\sigma \in G$, let:

$$D_\sigma = \{p' \in T \mid \sigma(p') \neq \pi(p')\}.$$

The condition (Γ) exactly expresses that D_σ is dense below p_0 . Hence, we can find a cofinal branch b through T such that $p_0 \in b$ and b meets every D_σ for $\sigma \in G$. As before, let $B = G[b]$. Again, B is a countable cover of T which respects G , by 1. Moreover, it seals π , because while $b \in B$, we know from b meeting D_σ that $\sigma[b] \neq \pi[b]$, and so $\pi[b] \notin B$. So B seals π , as desired.

4.) The proof here is similar to that of statement 3. To seal a potential additional branch f with domain T_p , we will use the fact that for every $\gamma < |T|$, the following set is dense below p , by clause (b) in the definition:

$$D_\gamma = \{r \mid \gamma \leq |f(r)|\}.$$

We may therefore choose a cofinal branch b through T meeting each of these countably many sets and containing p . Let c be the corresponding cofinal branch determined by $f[b]$. We now continue as in the proof of 3, using c in place of $\pi[b]$. In the case where there is a group G satisfying (Γ) with respect to f and p_0 , we modify the construction in the same way as in statement 3. That is, we construct the cofinal branch b to meet not only the previous D_γ , but also to meet the dense sets

$$D_\sigma = \{p' \in T \mid \sigma(p') \perp f(p')\},$$

for every $\sigma \in G$. Again, by 1, $B = G[b]$ is a countable set of branches covering T and respecting G . The set B seals f because $b \in B$ but the branch c determined by $f[b]$ is not in B .

5.) Suppose that f is a potential additional branch of some degree $n \in \omega$. So the domain of f is $T_{p_0} \times \cdots \times T_{p_{n-1}}$ for some $\vec{p} = \langle p_0, \dots, p_{n-1} \rangle \in T^n$, where $f(\vec{p}) \perp p_i$ for each $i < n$. We may find branches $b_0, \dots, b_n \in [T]$ with $p_i \in b_i$ such that $b_0 \times \cdots \times b_n$ meets each of the dense sets

$$D_\gamma = \{\vec{r} \in T^n \mid |f(\vec{r})| \geq \gamma\}.$$

This is possible because each D_γ is dense below \vec{p} in T^n . Let c be the branch determined by $f[b_0 \times \cdots \times b_{n-1}]$, and for each $q \in T$ choose a branch $b_q \in [T]$

such that $q \in b_q \neq c$. It follows that the set $B \stackrel{\text{def}}{=} \{b_q \mid q \in T\} \cup \{b_0, \dots, b_n\}$ covers T and seals f , since $c \notin B$.

6.) Suppose B is a countable set of branches through T and G is a countable group acting level transitively on T . For any branch $b \in B$ and $\sigma \in G$, the set

$$D_{\sigma,b} = \{p \in T \mid \sigma(p) \notin b\}$$

is dense in T . Choose a cofinal branch c through T meeting all $D_{\sigma,b}$ for $\sigma \in G$ and $b \in B$. By 1, the set $B' = G[c]$ is a countable set of branches covering T and respecting G . It is disjoint from B , by construction. \square

Let us introduce a class of automorphisms that we will use in our construction of a UBP tree that is not absolutely rigid. Our construction will involve certain γ -trees, subtrees of ${}^{<\gamma}2$, and we will consider the following class of automorphisms, which happen all to be restrictions of automorphisms of the full tree ${}^{\leq \gamma}2$. Specifically, for any ordinal γ and any $s \in {}^\gamma 2$, let π_s be the automorphism of ${}^{\leq \gamma}2$ that simply swaps the digits at the positive coordinates of s . More formally, define $\pi_s(t)$ by:

$$\pi_s(t)(\alpha) = \begin{cases} 1 - t(\alpha) & \text{if } \alpha \in \text{dom}(t) \text{ and } s(\alpha) = 1, \\ t(\alpha) & \text{if } \alpha \in \text{dom}(t) \text{ and } s(\alpha) = 0. \end{cases}$$

Let $\Pi^\gamma = \{\pi_s \mid s \in {}^\gamma 2\}$ be the corresponding group of such automorphisms. For $a \subseteq \gamma$, we shall write χ_a^γ to denote the characteristic function of a as a subset of γ , and we shall write π_a^γ for $\pi_{\chi_a^\gamma}$. Viewing Π^γ as a group of automorphisms, we shall write $\langle S \rangle$ for the subgroup of Π^γ generated by $S \subseteq \Pi^\gamma$. If G is a subgroup of Π^γ , and $p \in {}^{\leq \gamma}2$, we let $G[p] = \{\sigma(p) \mid \sigma \in G\}$ be the orbit of p under the canonical group action of Π^γ on ${}^{\leq \gamma}2$. We now record some simple but crucial facts about Π^γ .

Lemma 3.12 *Let γ be an ordinal.*

1. $\pi_a^\gamma \circ \pi_b^\gamma = \pi_{a \Delta b}^\gamma$, for $a, b \subseteq \gamma$.
2. Every element of Π^γ is self-inverse.
3. Π^γ is commutative, as a group of automorphisms.
4. Π^γ operates on itself via the group action assigning to each $\pi \in \Pi^\gamma$ the action $\pi_s \mapsto \pi_{\pi(s)}$. Denoting this action by “ \cdot ”, we have:

$$\pi_1 \cdot \pi_2 = \pi_1 \circ \pi_2,$$

for $\pi_1, \pi_2 \in \Pi^\gamma$. That is, it is the canonical group action of G on itself.

5. For any set $S \subseteq \Pi^\gamma$, the generated group $\langle S \rangle$ consists precisely of all automorphisms of the form $s_{i_0} \circ \dots \circ s_{i_{n-1}}$, where $\langle s_\alpha \mid \alpha < \kappa \rangle$ is a fixed enumeration of S and $i_0 < \dots < i_n < \kappa$. (The empty composition is taken here to be the identity.)

6. If G is a subgroup of Π^γ and $p \in {}^\gamma 2$, then

$$\langle \{\pi_q \mid q \in G[p]\} \rangle = \langle G \cup \{\pi_p\} \rangle.$$

Proof. These are routine verifications, although we give a proof of statement 6. Let $A \stackrel{\text{def}}{=} \langle \{\pi_q \mid q \in G[p]\} \rangle$ and $B \stackrel{\text{def}}{=} \langle G \cup \{\pi_p\} \rangle$. For the inclusion from left to right, assume that $\sigma \in A$. It follows by 5 that σ is a finite composition of automorphisms of the form $\pi_{\sigma_i(p)}$, which is equal to $\sigma_i \circ \pi_p$ by 4. Since the group is commutative, this composition has the form $\sigma_0 \circ \cdots \circ \sigma_{n-1} \circ (\pi_p)^n$, which is an element of B , as $(\pi_p)^n$ is either the identity or π_p , depending on whether n is even or odd. Conversely, every automorphism in B is of the form σ or $\sigma \circ \pi_p$, where $\sigma \in G$. In the latter case, $\sigma \circ \pi_p = \pi_{\sigma(p)} \in A$, and in the former, $\pi_p \in A$ because the identity is in G and $\sigma \circ \pi_p \in A$, so $\sigma \in A$. Hence the composition of these is in A again. But the composition is σ since π_p is self inverse. \square

We are finally ready to fulfill the third requirement of Observation 3.1, which we do under the assumption of \diamond .

Theorem 3.13 *If \diamond holds, then there is a Souslin tree with the unique branch property, but which is not absolutely rigid (and is absolutely non-rigid).*

Proof. Using \diamond , we will construct a 2-splitting Souslin tree with the unique branch property, but which is absolutely non-rigid. Our strategy will be to construct trees $T^{(n)}$ for $n < \omega$ with the following properties:

1. Each $T^{(n)}$ is a rigid 2-splitting Souslin tree.
2. Each $T^{(n)}$ has the unique branch property.
3. Forcing with any $T^{(n)}$ adds no cofinal branch to any other $T^{(m)}$.
4. Forcing with $T^{(n)}$ adds a nontrivial automorphism of $T^{(n+1)}$.

With such trees, we can build the final tree T by gluing together the trees $T^{(n)}$ as in figure 4. Since each $T^{(n)}$ is a 2-splitting Souslin tree, the resulting glued-together tree T is also a 2-splitting Souslin tree. Observe that forcing with T is equivalent to choosing some n and forcing with $T^{(n)}$. Since each $T^{(n)}$ has the unique branch property and no $T^{(n)}$ adds a branch to another $T^{(m)}$, this glued together tree T therefore has the unique branch property. Since forcing with $T^{(n)}$ adds an automorphism to $T^{(n+1)}$ and hence also to T , it follows that T is not absolutely rigid (and even absolutely non-rigid). Thus, the glued together tree T will satisfy all our desired properties, proving the theorem.

Let us begin the construction. We will construct the trees $T^{(n)}$ by simultaneous recursion on the levels. Each tree $T^{(n)}$ will consist of binary sequences, ordered by inclusion. We will inductively maintain that $T^{(n)}|_\alpha$ is a normal α -tree, in fact a subtree of ${}^{<\alpha}2$, ordered by inclusion. In addition, for any $\gamma < \alpha < \omega_1$ and $p \in T^{(n)}(\gamma)$, we will ensure that $\pi_p \upharpoonright (T^{(n+1)}|_{(\gamma+1)})$ is an automorphism of $T^{(n+1)}|_{(\gamma+1)}$, and the group generated by these automorphisms acts level-transitively on $T^{(n+1)}|_{(\gamma+1)}$.

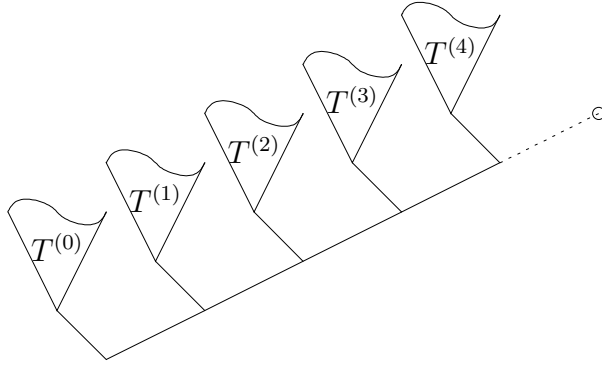


Figure 4: The trees $T^{(n)}$, glued together to form one composite tree.

Suppose that the trees $T^{(n)}|_\alpha$ have been constructed below α ; we must now specify the α^{th} levels $T^{(n)}(\alpha)$. At successor levels, there is no choice: we give every node on the top level of the tree two immediate successors by adjoining 0 and 1 to the binary sequences. This is easily seen to maintain our inductive assumptions. The interesting case occurs when α is a limit ordinal. In this case, our construction will always proceed with the following template. First, we will specify a particular set of branches B_0 covering $T^{(0)}|_\alpha$ and particular individual branches $b_n \in [T^{(n)}|_\alpha]$ for $n > 0$. This information will determine the α^{th} levels of the trees $T^{(n)}(\alpha)$ as follows. We let G_0 be the group generated by $\{\pi_b \mid b \in B_0\}$. Our induction hypothesis ensures that this group acts level transitively on $T^{(1)}|_\alpha$, and so the set $B_1 = G_0[b_1]$ is a covering set of branches for $T^{(1)}|_\alpha$, by Lemma 3.11.1. Continuing recursively, we define G_n to be the group of automorphisms generated by B_n , and $B_{n+1} = G_n[b_{n+1}]$. Inductively, G_n acts level transitively on $T^{(n+1)}|_\alpha$, and so B_{n+1} covers $T^{(n+1)}|_\alpha$. We now extend the trees to level α by defining $T^{(n)}(\alpha) = B_n$, conflating branches through ${}^{<\alpha}2$ with elements of ${}^\alpha 2$. Since B_{n+1} is the orbit of a single branch under G_n , it follows that G_n , which is the group generated by the elements of $T^{(n)}(\alpha) = B_n$, acts level transitively on $T^{(n+1)}(\alpha)$. Therefore, as long as our construction follows this pattern, we will preserve our induction hypotheses.

We therefore remain relatively free to choose the initial set of branches B_0 covering $T^{(0)}|_\alpha$ and the individual branches $b_n \in [T^{(n)}|_\alpha]$ for $n > 0$. We will do so in a way that will ensure that the trees $T^{(n)}$ are Souslin, that they each have the unique branch property and more generally, that forcing with $T^{(n)}$ will not add branches through any other $T^{(m)}$. In order to accomplish this, we will anticipate via \diamond the relevant potential additional branches and then seal them. So let us begin the detailed construction. Suppose that $\langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a \diamond sequence. We assume that the trees $T^{(n)}|_\alpha$ are defined up to the limit ordinal α ; we must now specify the branches $B_0 \subseteq [T^{(0)}|_\alpha]$ and the individual branches $b_n \in [T^{(n)}|_\alpha]$.

Case 0. We act first to ensure that the trees $T^{(n)}$ are Souslin. Suppose that D_α codes a set of the form $\langle 0, n, A \rangle$, where A is a maximal antichain in $T^{(n)}|_\alpha$. If $n = 0$, then we may choose a countable cover B_0 of $T^{(0)}|_\alpha$ that seals A , and choose $b_n \in [T^{(n)}|_\alpha]$ for $n > 0$ arbitrarily. Since our construction template leads to $T^{(0)}(\alpha) = B_0$, this will seal the antichain A below level α . So assume $n > 0$. In this case, we choose an arbitrary cover B_0 of $T^{(0)}|_\alpha$, and arbitrary branches b_1, \dots, b_{n-1} through $(T^{(1)}|_\alpha), \dots, (T^{(n-1)}|_\alpha)$, respectively. Let B_m and G_m be the resulting covering sets of branches and the corresponding groups in the construction template, for $m < n$. In particular, the group G_{n-1} acts level transitively on $T^{(n)}|_\alpha$. By Lemma 3.11.2, there is a branch $b_n \in [T^{(n)}|_\alpha]$ such that the corresponding set of branches $B_n = G_n[b_n]$ covers $T^{(n)}|_\alpha$, respects G_n and seals A . The remaining branches $b_k \in [T^{(k)}|_\alpha]$ for $k > n$ may be chosen arbitrarily, and we correspondingly define the α^{th} level of the trees according to the construction template. This procedure will ensure that the ultimate trees $T^{(n)}$ we construct will all be Souslin, because if $A \subseteq [T^{(n)}]$ is a maximal antichain, then there will be a stationary set of α for which D_α codes the triple $\langle 0, n, A \cap T^{(n)}|_\alpha \rangle$, at which point we will seal A below α . Therefore, it must be that $A = A \cap T^{(n)}|_\alpha$, and so the antichain was countable.

Case 1. Next, we act to ensure that the initial tree $T^{(0)}$ has the unique branch property. Suppose that D_α codes $\langle 1, f \rangle$, where f is a potential additional branch for $T^{(0)}|_\alpha$. By Lemma 3.11, we may choose a countable set of branches B_0 that covers $T^{(0)}|_\alpha$ and seals f . Choose branches $b_n \in [T^{(n)}|_\alpha]$ for $n > 0$ arbitrarily and follow the construction template to define the α^{th} level of the trees. Since this results in $T^{(0)}(\alpha) = B_0$, we have sealed f . It follows that the ultimate tree $T^{(0)}$ we construct will have the unique branch property, for if it did not, then there would be a potential additional branch f for $T^{(0)}$, and for a stationary set of α we would have that $f \upharpoonright (T^{(0)}|_\alpha)$ is a potential additional branch for $T^{(0)}|_\alpha$ and $D_\alpha = \langle 1, f \upharpoonright (T^{(0)}|_\alpha) \rangle$. At such a stage, we will have sealed $f \upharpoonright T^{(0)}|_\alpha$, meaning that there is a branch $b \in T^{(0)}(\alpha)$ such that $f[b]$ is cofinal in $T^{(0)}|_\alpha$ but has no upper bound in $T^{(0)}(\alpha)$. This contradicts that f was a potential additional branch on the entire tree $T^{(0)}$. So there can be no such potential additional branch, and the resulting tree $T^{(0)}$ will have the unique branch property.

Case 2. Next, we act to ensure that forcing with an earlier tree $T^{(m)}$ will not add a branch to a later tree $T^{(n)}$, for $m < n$. Suppose that D_α codes $\langle 2, m, n, f \rangle$, where $m < n$ and f is a potential additional branch for $(T^{(m)}|_\alpha) \sqcup (T^{(n)}|_\alpha)$ with $\text{dom}(f) \subseteq T^{(m)}|_\alpha$ and $\text{ran}(f) \subseteq T^{(n)}|_\alpha$. Choose a countable cover B_0 of $T^{(0)}|_\alpha$, taking care that if $m = 0$, then there is a $b_0 \in B_0$ such that $f[b_0]$ is cofinal in $T^{(n)}|_\alpha$. One can find such a branch by meeting countably many dense sets, since it is dense in $T^{(m)}|_\alpha$ that the values of $f(q)$ grow unbounded in α . Next, choose cofinal branches b_1, \dots, b_{n-1} through $T^{(1)}|_\alpha, \dots, T^{(n-1)}|_\alpha$, taking care that the branch b_m is chosen so that $f[b_m]$ is cofinal in $T^{(n)}|_\alpha$. Let B_1, \dots, B_{n-1} be the resulting covering sets of branches, with the associated groups G_1, \dots, G_{n-1} . We choose the next branch b_n with a bit more care. Namely, let c be the cofinal branch of $T^{(n)}|_\alpha$ generated by $f[b_m]$, and by 3.11.6, pick b_n in such a way that $G_{n-1}[b_n] \cap \{c\} = \emptyset$, thereby avoiding the branch c . It follows that the

induced set of branches $B_n = G_{n-1}[b_n]$ seals f , since b_m is in B_m and is hence extended, while the corresponding branch c , generated by $f[b_m]$, is not. Finally, we complete the construction by choosing the remaining branches b_k for $k > n$ arbitrarily, and building the corresponding α^{th} levels of the trees according to the construction template. This procedure will ensure that forcing with the ultimate tree $T^{(m)}$ will not add a branch to $T^{(n)}$, for if there were a $T^{(m)}$ -name τ for such a branch through $T^{(n)}$, then there would be a potential additional branch f with $\text{dom}(f) \subseteq T^{(m)}$ and $\text{ran}(f) \subseteq T^{(n)}$, and for a stationary set of α the restriction $f \upharpoonright (T^{(m)}|\alpha)$ would be a potential additional branch of the kind we considered in this case and the set D_α would code $\langle 2, m, n, f \upharpoonright (T^{(m)}|\alpha) \rangle$. At such a stage α , we would have sealed f below α by adding the branch b_m to $T^{(m)}(\alpha)$ but not extending $f[b_m]$ to any node in $T^{(n)}(\alpha)$, contradicting the fact that f was a potential additional branch.

Case 3. Next, we act to ensure that forcing with a later tree $T^{(m)}$ will not add a branch to an earlier tree $T^{(n)}$, for $n < m$. Suppose that D_α codes $\langle 3, m, n, f \rangle$, where $n < m$ and f is a potential additional branch for $(T^{(m)}|\alpha) \sqcup (T^{(n)}|\alpha)$ with $\text{dom}(f) = (T^{(m)}|\alpha)_{\bar{p}}$ and $\text{ran}(f) \subseteq T^{(n)}|\alpha$. Choose any set of branches B_0 covering $T^{(0)}|\alpha$, and any branches b_1, \dots, b_{m-1} cofinal in $(T^{(1)}|\alpha), \dots, (T^{(m-1)}|\alpha)$, respectively. Let B_1, \dots, B_{m-1} and G_0, \dots, G_{m-1} be the corresponding sets of branches and automorphism groups resulting from our construction template. We choose the next branch $b_m \in [T^{(m)}|\alpha]$ in such a way so as to seal f . For any $\sigma \in G_{m-1}$, $c \in B_n$ and $\gamma < \alpha$, let

$$D_{\sigma, c, \gamma} = \left\{ q \in (T^{(m)}|\alpha) \mid \begin{array}{l} \gamma \leq |q|, \text{ and if } \sigma(q) \in \text{dom}(f), \\ \text{then } f(\sigma(q)) \perp c \text{ and } |f(\sigma(q))| \geq \gamma \end{array} \right\}.$$

We argue that this set is dense in $T^{(m)}|\bar{\alpha}$. Given any $\bar{p} \in T^{(m)}|\bar{\alpha}$, choose an extension p of \bar{p} such that $|p| \geq \gamma$. If $p \in D_{\sigma, c, \gamma}$, we're done. Otherwise, since f is a potential additional branch and $\sigma(p) \in \text{dom}(f)$, there is a node $p' \geq \sigma(p)$, such that $|f(p')| \geq \gamma$. Further, there are extensions r_0 and r_1 of $\sigma(p')$ in $T^{(m)}$ such that $f(r_0) \perp f(r_1)$. We may assume $f(r_0) \perp c$. Let $q = \sigma^{-1}(r_0)$ (which is the same as $\sigma(r_0)$), and observe that $q \in D_{\sigma, c, \gamma}$ and $q > p \geq \bar{p}$. So $D_{\sigma, c, \gamma}$ is indeed dense.

Continuing with the construction, we now choose b_m through $T^{(m)}|\alpha$ so as to meet every $D_{\sigma, c, \gamma}$. Choose b_k cofinal in $T^{(k)}|\alpha$ for $k > m$ arbitrarily, and carry out the construction template to define the α^{th} level of all of the trees. We will argue that f is sealed by the resulting $T^{(m)}(\alpha) = B_m = \{\sigma[b] \mid \sigma \in G_{m-1}\}$. Suppose that $b \in B_m$ and $\bar{p} \leq b$. The branch b has the form $b = \sigma[b_m]$ for some $\sigma \in G_{m-1}$. Since b_m meets every $D_{\sigma, c, \gamma}$ for $c \in B_n$, it follows that $f[\sigma(b_m)] \perp c$. Thus, $f[b]$ is not extended by any branch in B_n , and so we have sealed f . This procedure in our construction therefore ensures that forcing with the ultimate tree $T^{(m)}$ will not add a cofinal branch to $T^{(n)}$, because if there were a $T^{(m)}$ -name for such a branch, then there would be a potential additional branch function f , which would be anticipated and sealed at some stage α .

Case 4. Lastly, we act to ensure that every tree $T^{(n)}$, for $n > 0$, has the unique branch property. This case is more complicated than the earlier cases.

Suppose that D_α codes $\langle 4, n, f \rangle$, where $n > 0$ and f is a potential additional branch for $T^{(n)}|_\alpha$. We will choose a set of branches $B_0 = \{d_k \mid k < \omega\}$ covering $T^{(0)}|_\alpha$ and individual branches $b_m \in [T^{(m)}|_\alpha]$ for $m > 0$ in such a way that, after following the construction template, the resulting tree $T^{(n)}(\alpha)$ seals f . Choosing these branches amounts to choosing a filter in the following poset \mathbb{P} , defined with finite support in each factor:

$$\mathbb{P} = (T^{(0)}|_\alpha)^\omega \times \prod_{m>0} T^{(m)}|_\alpha.$$

Such a filter $H \subseteq \mathbb{P}$ determines $B_0 = \{d_k \mid k < \omega\}$ and b_m for $m > 0$ by simply projecting onto the appropriate coordinates, so that d_k is the projection of H onto the k^{th} coordinate of the first factor of \mathbb{P} and b_m is the projection onto the m^{th} coordinate of the second factor. We will construct H to have the desired properties by meeting countably many dense sets in \mathbb{P} . First, it is easy to see that with countably many dense sets in \mathbb{P} we can ensure that the resulting B_0 is a covering set of cofinal branches through $T^{(0)}|_\alpha$ and that each b_m is a cofinal branch in $T^{(m)}|_\alpha$. The reader can verify that an additional list of dense sets will ensure that $f[b_n]$ is cofinal in $T^{(n)}|_\alpha$. In order to seal f , what we want to do is ensure that we do not add a branch to B_n extending $f[b_n]$. Of course, B_n is determined by b_n and G_{n-1} , which is determined by the earlier choices of b_m and so on, and ultimately also by B_0 .

Let us briefly analyze in greater detail the groups G_m arising in the construction template. We claim that each G_m is the group generated by G_0 and the individual automorphisms $\pi_{b_1}, \dots, \pi_{b_m}$. We prove this by induction on m . For the anchor case $m = 0$, there is nothing to show. Assume inductively that the claim holds for m , and consider $G_{m+1} = \{\pi_b \mid b \in B_{m+1}\} = \{\pi_b \mid b \in G_m[b_{m+1}]\}$. By Lemma 3.12, item 6, it follows that $G_{m+1} = \langle G_m \cup \{\pi_{b_{m+1}}\} \rangle$. By induction, G_m is generated by G_0 and $\pi_{b_1}, \dots, \pi_{b_m}$, so it now follows that G_{m+1} is generated by G_0 and $\pi_{b_1}, \dots, \pi_{b_m}, \pi_{b_{m+1}}$, as we claimed. It follows that $T^{(n)}(\alpha) = B_n$ will be the same as $\langle G_0 \cup \{\pi_{b_1}, \dots, \pi_{b_{n-1}}\} \rangle[b_n]$.

We now continue with the construction. We want to ensure that $f[b_n]$ is not extended to any branch in B_n . Thus, it will suffice to ensure that $f[b_n]$ is incompatible with $\sigma[b_n]$ for every $\sigma \in \langle G_0 \cup \{\pi_{b_1}, \dots, \pi_{b_{n-1}}\} \rangle$. Any such σ has the form $\sigma = (\pi_{d_{i_1}} \circ \dots \circ \pi_{d_{i_u}}) \circ (\pi_{b_{j_1}} \circ \dots \circ \pi_{b_{j_v}})$, where $a = \{i_1, \dots, i_u\} \subseteq \omega$ and $b = \{j_1, \dots, j_v\} \subseteq \{1, \dots, n-1\}$ (allowing a or b to be empty). The conditions in \mathbb{P} provide partial information about the corresponding branches \vec{d} and \vec{b} , and we will choose the filter H in such a way so as to ensure that $\sigma[b_n] \perp f[b_n]$. For notational convenience let us regard elements $p \in \mathbb{P}$ as having the form $p = (\vec{p}, \vec{q})$, where $\vec{p} = \langle p_0, p_1, \dots \rangle$ and $\vec{q} = \langle q_1, q_2, \dots \rangle$, with $p_i \in (T^{(0)}|_\alpha)$ providing partial information about d_i and $q_j \in (T^{(j)}|_\alpha)$ providing partial information about b_j . Note that all but finitely many p_i 's and q_j 's will be the root node of the corresponding tree, since the product \mathbb{P} used finite support.

For any such condition $p = (\vec{p}, \vec{q})$ and any $a = \{i_1, \dots, i_u\} \subseteq \omega$ and $b = \{j_1, \dots, j_v\} \subseteq \{1, \dots, n-1\}$, let $\pi_{\vec{p}, \vec{q}}^{a,b}$ be the corresponding partial par-

tial automorphism

$$\pi_{\vec{p}, \vec{q}}^{a,b} \stackrel{\text{def}}{=} (\pi_{p_{i_1}} \circ \cdots \circ \pi_{p_{i_u}}) \circ (\pi_{q_{j_1}} \circ \cdots \circ \pi_{q_{j_v}}).$$

Such a partial automorphism is exactly what will grow into a $\sigma \in G_{n-1}$ as we explained above. In the following, we shall allow ourselves to apply π_s to t even if $|t| > |s|$, by defining $\pi_s(t) = \pi_s(t \upharpoonright |s|)$. It follows that if $s \subseteq s'$, then $\pi_s(t) \subseteq \pi_{s'}(t)$. Let $\text{dom}(f) = (T^{(n)}|_\alpha)_{\vec{p}}$. We claim now that for any $a = \{i_1, \dots, i_u\} \subseteq \omega$ and $b = \{j_1, \dots, j_v\} \subseteq \{1, \dots, n-1\}$, the following set $D_{a,b}$ is dense in \mathbb{P} .

$$D_{a,b} \stackrel{\text{def}}{=} \{ (\vec{p}, \vec{q}) \in \mathbb{P} \mid q_n \perp \vec{p} \text{ or } \pi_{\vec{p}, \vec{q}}^{a,b}(q_n) \perp f(q_n) \}$$

If a and b are both empty, then the claim follows from the fact that $\vec{p} \perp f(\vec{p})$, as f is a potential additional branch. If a or b is nonempty, then any change to any p_i or q_j appearing in $a \cup b$ at an ordinal below $|q_n|$ will cause a change in $\pi_{\vec{p}, \vec{q}}^{a,b}(q_n)$ at the corresponding ordinal. Thus, some of these extensions will exhibit incompatibility with $f(q_n)$, putting them in $D_{a,b}$. More specifically, suppose $p = (\vec{p}, \vec{q}) \in \mathbb{P}$ is given. If $q_n \perp \vec{p}$, then we're done; so assume the contrary. Suppose a is nonempty, and fix some $i \in a$ (the case where b is nonempty is similar). Let $\delta = |p_i|$. By extending p to a stronger condition, we may assume that p_i is the shortest sequence, that is, the lowest node, appearing in p , so that all other p_k and q_l are defined at δ . Similarly, by extending q_n inside \vec{q} , we may assume that $|f(q_n)|$ is also larger than δ . Now build two extensions r and s of p by extending p_i to add either 0 or 1. That is, $r = (\vec{r}, \vec{q})$ and $s = (\vec{s}, \vec{q})$, where $r_i = p_i \frown \langle 0 \rangle$ and $s_i = p_i \frown \langle 1 \rangle$, and otherwise $r_k = s_k = p_k$ for $k \neq i$. Because we made just one change, at δ , it follows that $\pi_r^{a,b}(q_n)$ and $\pi_s^{a,b}(q_n)$ differ at δ . Thus, they cannot both be compatible with $f(q_n)$, and so r or s must be in $D_{a,b}$. So $D_{a,b}$ is dense, as we claimed.

We now complete the argument in this case. Let H be a filter in \mathbb{P} meeting all the dense sets we have mentioned, such that \vec{p} is in the projection of H onto the n^{th} coordinate of the second factor of \mathbb{P} (so that the fact that it meets $D_{a,b}$ is meaningful). Let $B_0 = \{d_k \mid k < \omega\}$ and b_m for $m > 0$ be the resulting branches, obtained by projecting H . By construction, B_0 covers $T^{(0)}|_\alpha$ and b_m is cofinal in $T^{(m)}|_\alpha$. Furthermore, any $\sigma \in G_{n-1}$ is determined, as we explained, by some a and b as above (so that $\sigma = \pi_{\vec{a}, \vec{b}}^{a,b}$), and since H meets $D_{a,b}$ we ensured that $\sigma[b_n] \perp f[b_n]$. Thus, we have $b_n \in B_n$ but $f[b_n]$ is not extended by any element of B_n . Therefore, our construction template ensures that f is sealed below α . It follows that the ultimate tree $T^{(n)}$ we construct will have the unique branch property, since as in the other cases, any putative potential additional branch reflects to a stage α , where it was sealed.

We have therefore constructed the trees $T^{(n)}$ to have the four features we claimed at the beginning of this proof. So the proof of Theorem 3.13 is now complete. \square

With Theorems 3.2, 3.4 and 3.13, we have now fulfilled the three requirements of Observation 3.1, using \diamond . Consequently, our main result, Main Theorem 1.1, is now proved.

4 A larger context of rigidity

The implication diagram of Figure 1 is part of a larger implication diagram, pictured below, featuring the other rigidity notions we have considered. The diagram continues to the right by considering n -absolute forms of the rigidity notions, leading up to $<\omega$ -absolute rigidity notions or more.

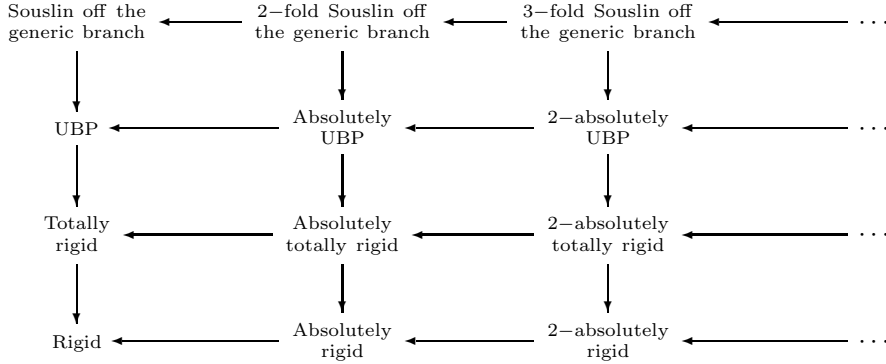


Figure 5: Larger implication diagram

Question 4.1 *Is the implication diagram of Figure 5 complete?*

It seems quite possible to modify our \diamond constructions so as to ensure that the resulting tree is Souslin off the generic branch, by simply anticipating and sealing the potential antichains. Conversely, to construct a tree that is UBP but not Souslin off the generic branch, one idea would be to make a Souslin tree such that adding a branch through it creates a specializing function for some other part of the tree. Beyond this, it seems difficult to separate Souslin off the generic branch from absolutely UBP. We leave such problems for the future.

5 An application to the automorphism tower problem

In this final section, we present an application of our methods to a question involving the automorphism tower problem in group theory, an application which was the original motivation of our investigation. The issue is that while the main theorem of [HT00] showed that it is consistent with ZFC that there is

a group whose automorphism tower is highly malleable by forcing, it was not known whether such a group exists in all models of set theory or, for example, in the constructible universe L . The specific group G constructed in [HT00] was obtained by first forcing to add several V -generic Souslin trees and then constructing G from a certain subgroup of the automorphism group of a certain graph built from these trees. The strong rigidity properties of the generic Souslin trees led to the desired malleability of the automorphism tower of this group. Here, we use \diamond to construct such Souslin trees with the desired rigidity properties. Consequently, in L there are such groups whose automorphism towers are highly malleable by forcing.

We quickly review the automorphism tower construction in group theory. If G is a group, then so is the automorphism group $\text{Aut}(G)$, the set of isomorphisms of G to itself, and there is a natural homomorphism of G into $\text{Aut}(G)$ arising from conjugation. Specifically, every element $g \in G$ maps to the corresponding inner automorphism $i_g : h \mapsto ghg^{-1}$. The automorphism tower of G is obtained by iterating this process transfinitely.

$$G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_\alpha \rightarrow G_{\alpha+1} \rightarrow \cdots$$

One begins with $G_0 = G$ and uses the canonical homomorphism of G_α into $G_{\alpha+1} = \text{Aut}(G_\alpha)$ at successor steps. At any limit stage, G_λ is the direct limit of the previous groups G_α , for $\alpha < \lambda$, with respect to these natural homomorphisms. The automorphism tower *terminates* if it eventually reaches a fixed point, that is, if for some α , the canonical map from G_α to $G_{\alpha+1}$ is an isomorphism. This occurs if G_α is a *complete* group, a centerless group having only inner automorphism. The *height* of the tower is the least α for which this occurs. If G is centerless, then one can show that all the groups G_α in the tower are centerless, and consequently all the maps $G_\alpha \rightarrow G_{\alpha+1}$ are injective. In this case, therefore, one can identify each group with its image and view the tower as building up larger and larger groups, with direct limits corresponding simply to unions. Wielandt [Wie39] proved the classical result that the automorphism tower of any centerless finite group terminates in finitely many steps. Later, various authors [RR70], [Hul70] proved that larger classes of centerless groups had terminating automorphism towers. Simon Thomas [Tho85], [Tho98] solved the automorphism tower problem for centerless groups by proving that every centerless group has a terminating automorphism tower. Building on this, the second author [Ham98] proved that every group has a terminating automorphism tower. The main theorem of [HT00] revealed that the automorphism tower problem has what might be termed a set theoretic essence, namely, the fact that the automorphism tower of a group can be sensitive to the model of set theory in which it is computed. We state a special case of this phenomenon here:

Theorem 5.1 ([HT00]) *For any $n < \omega$, there is a forcing extension with a group G , whose automorphism tower has height exactly n , but for any nonzero $m < \omega$, there is a further (c.c.c.) forcing extension in which the automorphism tower of the very same group G has height m .*

Thus, the automorphism tower of G is sensitive to the set theoretic background, and even the height of the automorphism tower of G can be precisely controlled, becoming shorter or taller in various forcing extensions, as desired. The actual theorem proved in [HT00] is stronger than stated above and is obtained by replacing ω with an arbitrary ordinal λ ; indeed, [HT00] shows that it is consistent to have a proper class of such groups for all ordinals λ at once. The open question with these theorems was whether one must force to add the group G . Perhaps one can prove in **ZFC** that there are such groups? Are there such groups in the constructible universe L ? We answer at least this latter question by proving that the combinatorial principle \diamond , which holds in L , is sufficient to construct the groups.

Theorem 5.2 *Assume \diamond holds. Then for every $n < \omega$ there is a group G , whose automorphism tower has height n , but for any nonzero $m < \omega$ there is a (c.c.c.) forcing extension in which the automorphism tower of G has height m .*

Let us state without much explanation that the main group-theoretic construction of [HT00] shows that Theorem 5.2 is a consequence of the following combinatorial Theorem 5.3, which we will prove. The group G of Theorem 5.2 is closely connected with a certain subgroup of the automorphism group of a graph containing multiple copies of the trees T^n of Theorem 5.3. The crucial property 3 of Theorem 5.3 allows the automorphism group of this graph to be precisely controlled by forcing. For the details of this construction, we refer the readers to [HT00], and to the survey article [Ham01], which contains a gentle overview of it. Thomas' forthcoming monograph [Tho] includes extensive discussion of this construction and many other aspects of the automorphism tower problem.

Theorem 5.3 *Assume \diamond holds. Then there is an infinite sequence of Souslin trees T^n with the following properties:*

1. *Each T^n is a rigid Souslin tree.*
2. *The trees T^n are pairwise non-isomorphic.*
3. *For any $m < \omega$, there is a c.c.c., countably distributive forcing extension preserving the rigidity of all of the trees, in which $T^0 \cong \dots \cong T^m$, but $T^i \not\cong T^j$ if $m \leq i < j$.*

Proof. Suppose that $\vec{D} = \langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a \diamond -sequence. We will define the trees T^n by recursively defining their levels $T^n(\alpha)$. Simultaneously, we will recursively define certain controller trees $T^{n,m}$, for every $n < m < \omega$, which will be used when it is desired to force the existence of an isomorphism from T^n to T^m . All of these trees will be rigid Souslin trees. The difficulty will be to ensure that forcing with a finite product of the controller trees, in order to force $T^0 \cong \dots \cong T^m$ as in statement 3 of Theorem 5.3, will not create unwanted automorphisms of any T^n or unwanted isomorphisms from T^i to T^j for $m \leq i < j$.

We will construct the trees T^n and $T^{n,m}$ so as to inductively maintain the following properties, where π_s is the automorphism swapping the values at the positive coordinates of s , as in Lemma 3.12.

1. Each $T^n|_\alpha$ and $T^{n,m}|_\alpha$ is a normal α -tree, a subtree of ${}^{<\alpha}2$.
2. If $s \in T^{n,m}(\alpha)$, then $\pi_s \upharpoonright (T^n|_\alpha)$ is an isomorphism of $T^n|_\alpha$ with $T^m|_\alpha$.
3. These actions are level transitive in the following sense: if $p \in T^n(\alpha)$ and $q \in T^m(\alpha)$, then there is a finite list n_0, n_1, \dots, n_k leading from $n_0 = n$ to $n_k = m$, and $s_i \in T^{n_i, n_{i+1}}(\alpha)$ such that $\pi_{s_{k-1}} \circ \dots \circ \pi_{s_0}(p) = q$. Here, we let $T^{m,n} \stackrel{\text{def}}{=} T^{n,m}$ in case $m > n$. Note that this makes sense, since every π_s , coming from $s \in T^{n,m}$, is also an isomorphism from T^m to T^n , since it is self-inverse.

All the trees begin, of course, with the empty root node $\langle \rangle$, and at successor stages we extend every node on the α^{th} level with its two immediate successors in ${}^\alpha 2$. It is easy to see that this maintains our inductive assumptions. If λ is a limit ordinal and the trees are defined at levels below λ , then we will always define the λ^{th} level by specifying for each pair n, m a covering set of branches $B^{n,m} \subseteq [T^{n,m}|_\lambda]$ and, for one specific n_0 , a path $b \in [T^{n_0}|_\lambda]$. We will then define the λ^{th} level of the controller trees to be $T^{n,m}(\lambda) = B^{n,m}$ and the λ^{th} level $T^n(\lambda)$ to consist of all images $\pi_{\vec{s}}(b)$ under the resulting compositions $\pi_{\vec{s}} = \pi_{s_{n-1}} \circ \dots \circ \pi_{s_0}$ leading from $T^{n_0}|_\lambda$ to $T^n|_\lambda$, with $s_i \in B^{n_i, n_{i+1}}$, as in our inductive assumption 3 above.

We claim that as long as our recursive construction follows this pattern, then we will maintain our inductive assumptions. To see this, observe first that the tree $T^{n,m}|_{(\lambda+1)}$ will be normal, because $B^{n,m}$ covers $T^{n,m}|_\lambda$. The tree $T^n|_{(\lambda+1)}$ will be normal because the level-transitive action of the automorphisms ensures that the images of b cover every tree. Every $s \in T^{n,m}(\lambda)$ will provide an isomorphism π_s of $T^n|_{(\lambda+1)}$ to $T^m|_{(\lambda+1)}$ since we explicitly closed $T^m(\lambda)$ under the images of such π_s . And finally, the action will still be level-transitive, because we define the level $T^n(\lambda)$ to be the images of the single node b under all the compositions of isomorphisms. So, given $\pi_{\vec{s}}(b)$ and $\pi_{\vec{t}}(b)$, we can let \vec{s}^{-1} be the path \vec{s} reversed, and let $\vec{u} = \vec{s}^{-1} \frown \vec{t}$ be the composition of the paths, leading from where \vec{s} led to where \vec{t} leads. Then, $\pi_{\vec{t}}(b) = \pi_{\vec{u}}(\pi_{\vec{s}}(b))$, as wished. So as in Theorem 3.13 we are relatively free to choose the branches $B^{n,m} \subseteq [T^{n,m}|_\lambda]$ and the branch $b \in [T^{n_0}|_\lambda]$ so as to ensure that all the trees are Souslin, rigid, mutually rigid, and so on. For this, we will use the diamond sequence \vec{D} as in our earlier constructions to anticipate and then seal any unwanted antichains or automorphisms or potential automorphisms that might arise. So let us now explain the particular details of how we choose $B^{n,m}$, n_0 and b at level λ .

We begin with the easiest case, where we act to ensure that every T^n is Souslin. Suppose that D_λ codes the triple $\langle 0, n_0, A \rangle$, where n_0 is a natural number and A is a maximal antichain in $T^{n_0}|_\lambda$. In this case, extend the controller

trees by specifying any covering set of branches $B^{n,m}$, and let $G_\lambda^{n_0}$ be the corresponding group of automorphisms of $T^{n_0}|\lambda$ obtained by the compositions $\pi_{\vec{s}}$ leading from n_0 to n_0 . By Lemma 3.11, we may find a branch $b \in [T^{n_0}|\lambda]$ so that the resulting set of branches $G_\lambda^{n_0}[b]$ seals A . This procedure will ultimately ensure that the tree T^{n_0} is Souslin, because for any antichain $A \subseteq T^{n_0}$, there would be a stationary set of α such that D_α codes $\langle 0, n_0, A \cap T^{n_0}|\alpha \rangle$, at which point A would be sealed below level α . Therefore, the ultimate tree T^{n_0} could have no uncountable antichain.

Similarly, we act to seal antichains in the controller trees $T^{n,m}$. Specifically, if D_λ codes $\langle 1, n, m, A \rangle$ where $A \subseteq T^{n,m}|\lambda$ is a maximal antichain, then we choose the branches $B^{n,m}$ so as to seal A , and then choose the other branches $B^{k,r}$, n_0 and $b \in [T^{n_0}|\lambda]$ arbitrarily. The result is that A is sealed in $T^{n,m}$ and so every controller tree $T^{n,m}$ will be Souslin. In this case, the choice of n_0 is irrelevant.

More generally, we act next to ensure that the finite product forcing $T^{0,\dots,m} = T^{0,1} \times \dots \times T^{m-1,m}$ will have the countable chain condition. Specifically, suppose that D_λ codes $\langle 2, m, A \rangle$, where A is a maximal antichain in $T^{0,\dots,m}|\lambda$. We will build covering sets of branches $B^{i,i+1} \subseteq [T^{i,i+1}|\lambda]$ for $i < m$ to fulfill the requirement that the corresponding products $\vec{b} = \langle b_0, \dots, b_{m-1} \rangle$ with $b_i \in B^{i,i+1}$ all lie above a node in A . To do this, consider the partial order

$$\mathbb{P} = (T^{0,1}|\lambda)^\omega \times \dots \times (T^{m-1,m})^\omega$$

with finite support in every factor. Let us view elements of \mathbb{P} as vectors $\vec{q} = \langle q^0, \dots, q^{m-1} \rangle$, where each q^i is a function from ω to $T^{i,i+1}$. We will construct a pseudo-generic filter G in \mathbb{P} by meeting certain dense sets, and then construct the covers $B_{i,i+1}$, for $i < m$, by setting:

$$B_{i,i+1} = \left\{ \bigcup \{ q^i(k) \mid \langle q^0, \dots, q^{m-1} \rangle \in G \} \mid k < \omega \right\}.$$

We ensure that $B_{l,l+1}$ covers $T^{l,l+1}|\lambda$ by meeting the following dense sets, for every $p \in T^{l,l+1}|\lambda$ and $l < m$.

$$D_{p,l} = \{ \langle q^0, \dots, q^{m-1} \rangle \in \mathbb{P} \mid \exists k < \omega \quad p \leq q^l(k) \},$$

We ensure that A is sealed by meeting the following dense sets, for every $\vec{l} \in \omega^m$.

$$D_{\vec{l}} = \{ \langle q^0, \dots, q^{m-1} \rangle \in \mathbb{P} \mid \exists r \in A \quad \langle q^0, \dots, q^{m-1} \rangle \geq r \},$$

And we ensure that the new branches are cofinal by meeting the following dense sets, for every $i < m$, $j < \omega$ and $\alpha < \lambda$.

$$D_{i,j,\gamma} = \{ \langle q^0, \dots, q^{m-1} \rangle \in \mathbb{P} \mid |q^i(j)| \geq \alpha \},$$

After this, for the other values of n and k , we let $B^{n,k} \subseteq [T^{n,k}|\lambda]$ be any covering set of branches and choose n_0 and $b \in [T^{n_0}|\lambda]$ arbitrarily to complete the definition of the λ^{th} level as earlier. This part of the construction will ensure

that the ultimate $T^{0,\dots,m}$ is c.c.c., because any maximal antichain $A \subseteq T^{0,\dots,m}$ will be anticipated by some D_λ , and sealed at stage λ as above. So all such antichains are bounded.

Next, we act to ensure that all the trees T^n are rigid. For convenience, we will ensure that they have the unique branch property. Suppose D_λ codes $\langle 2, n_0, f \rangle$, where f is a potential additional branch for $T^{n_0}|\lambda$. In this case, we extend the controller trees by specifying any covering set of branches $B^{n,m}$ and let $G_\lambda^{n_0}$ be the corresponding group of automorphisms $\pi_{\vec{s}}$ of $T^{n_0}|\lambda$. By Lemma 3.11, we may find a branch $b \in [T^{n_0}|\lambda]$ such that corresponding generated set of branches $G_\lambda^{n_0}[b]$ seals f . It follows that T^{n_0} will have the unique branch property (and hence be rigid), since we will have killed off any potential additional branch.

Similarly, we can also ensure that each controller tree $T^{n,m}$ has the unique branch property. If D_λ codes $\langle 3, n, m, f \rangle$, where f is a potential additional branch of $T^{n,m}|\lambda$, then we simply choose the branches $B^{n,m}$ so as to seal f , and then choose n_0 and $b \in [T^{n_0}]$ arbitrarily. This kills off any potential additional branch for $T^{n,m}$, and so it will have the unique branch property.

Let us now observe some facts that are already determined about the trees we will ultimately construct. Forcing with the controller tree $T^{n,m}$ produces a generic cofinal branch $s \in [T^{n,m}]$. By the inductive assumption 3, the corresponding automorphism π_s of ${}^{<\omega_1}2$ takes T^n to T^m . Thus, π_s shows that $T^n \cong T^m$ in the forcing extension $V[s]$. So forcing with the n -fold product $T^{0,1,\dots,m} = T^{0,1} \times \dots \times T^{m-1,m}$ will ensure $T^0 \cong T^1 \cong \dots \cong T^m$ in the extension $V[s_0, \dots, s_n]$. In order to prove the theorem, therefore, we must ensure that the trees T^n exhibit no other unwanted isomorphism relations. In the next case of our recursive construction, therefore, we will anticipate and seal any such unwanted potential isomorphisms.

Specifically, suppose that D_λ codes $\langle 5, m, i, j, f \rangle$, where $m \leq i < j$ and f is a $(T^{0,1,\dots,m}|\lambda)$ -potential isomorphism of $T^i|\lambda$ to $T^j|\lambda$. That is, f is a function with domain $T^{0,1,\dots,m}|\lambda$, such that $f(\vec{q})$ is a partial isomorphism of $T^i|\lambda$ to $T^j|\lambda$, and for any condition $\vec{q} \in T^{0,1,\dots,m}|\lambda$, there is a dense set of stronger conditions \vec{r} , whose $f(\vec{r})$ extends $f(\vec{q})$ so as to insert any given node of $T^i|\lambda$ into the domain and any given node of $T^j|\lambda$ into the range. Such functions f exactly arise from $T^{0,1,\dots,m}$ -names for isomorphisms of T^i to T^j . We will now seal this potential isomorphism. First, by a simple diagonalization meeting countably many dense sets, we choose branches $\vec{b} = \langle b_0, \dots, b_{m-1} \rangle$ with $b_k \in [T^{k,k+1}|\lambda]$ in such a way that $f[\vec{b}] = \bigcup \{f(\vec{p}) \mid \vec{p} \leq \vec{b}\}$ is an isomorphism of $T^i|\lambda$ to $T^j|\lambda$. According to our construction pattern, we have to specify n_0 and a branch b through $T^{n_0}|\lambda$. Set $n_0 = i$ and fix an arbitrary cofinal branch $b \in [T^i|\lambda]$. Next, we will choose further covering sets $B^{n,k}$ of branches for each tree $T^{n,k}|\lambda$ in such a way so as to guarantee the following crucial property: for any finite list n_0, \dots, n_k from $n_0 = i$ to $n_k = j$ and any $\vec{s} = \langle s_0, \dots, s_k \rangle$ with $s_r \in B^{n_r, n_{r+1}}$ we have $\pi_{\vec{s}}(b) \neq f[\vec{b}][b]$. If we can do this, then the corresponding definition as above, of $T^{n,k}(\lambda) = B^{n,k}$ and $T^n(\lambda)$ as the set of all the relevant $\pi_{\vec{t}}(b)$, will kill off $f[\vec{b}]$ as an isomorphism of T^i to T^j , since the branch b was extended in T^i but its image $f[\vec{b}][b]$ was not extended in T^j . The key observation is that

we can find the branches $B^{n,k}$ with the crucial property that we mentioned by meeting countably many dense sets. Specifically, we will associate to each node $p \in T^{n,k}|\lambda$ a branch $b_p \in [T^{n,k}|\lambda]$, determined by our specifying longer and longer initial segments of it. For each of the countably-many possible patterns n_0, \dots, n_k leading from i to j and the possible s_0, \dots, s_k chosen from the b_p we are specifying, we look at the partial information about $\pi_{\vec{s}}(b)$ that has been determined by the initial segments of \vec{s} . Since the initial branches b_0, \dots, b_m that decide f do not involve a controller of T^j , it must be at any stage of the construction that at least one of the branches in such an \vec{s} is only partially specified. Extending this branch in incompatible ways leads to incompatible isomorphisms $\pi_{\vec{s}}$, and so in particular, by making a greater commitment to the partially specified branch b_p in \vec{s} , we can ensure that the corresponding $\pi_{\vec{s}}$ acts on b in a way that is different from $f[\vec{b}][b]$. By enumerating these requirements in an ω -sequence, we can systematically meet them all, and thereby build the sets of covering branches as desired. It now follows that forcing with $T^{0,\dots,m}$ will not create unwanted isomorphisms from T^i to T^j when $m \leq i < j$.

Finally, a similar method can be used to ensure that forcing with $T^{0,\dots,m}$ does not add automorphisms of any T^i . Specifically, if there were a $T^{0,\dots,m}$ -name for a new automorphism of T^i , then there would be a $T^{0,\dots,m}$ -potential automorphism of T^i , a function f mapping $T^{0,\dots,m}$ to partial automorphisms of T^i , in such a way that one can add nodes to the domain or range of $f(\vec{q})$ by strengthening \vec{q} to \vec{r} in $T^{0,\dots,m}$ and considering $f(\vec{r})$. If D_λ codes $\langle 6, m, i, f \rangle$ where f is such a $(T^{0,\dots,m}|\lambda)$ -potential automorphism of $T^i|\lambda$, then we can seal f as follows: In our construction pattern, again let $n_0 = i$. Fix a branch $b \in [T^i|\lambda]$ and branches $\vec{b} = \langle b_0, \dots, b_{m-1} \rangle$, where $b_i \in [T^{i,i+1}|\lambda]$ in such a way that $f[\vec{b}]$ is a nontrivial automorphism of $T^i|\lambda$. Now find covering branches $B^{n,k}$ for $T^{n,k}|\lambda$ and a branch as above, by specifying longer and longer initial segments, so that $f[\vec{b}][b] \neq \pi_{\vec{s}}(b)$ for any $\vec{s} = \langle s_0, \dots, s_m \rangle$ leading from i to i . The key point again is that any list n_0, \dots, n_k leading from i to i with nontrivial \vec{s} and $s_r \in B^{n_r, n_{r+1}}$ will necessarily involve at least one branch s_r that is only partially specified, since one cannot use the initial branches b_0, \dots, b_m alone to form such a closed loop from i to i without repeating the branches (which would then cancel as the π_s commute and are self-inverse).

This completes the recursive construction of the trees T^n and the controller trees $T^{n,m}$. By design, any branch s through $T^{n,m}$ will add an isomorphism π_s from T^n to T^m , and so forcing with $T^{0,\dots,m}$ ensures $T^0 \cong \dots \cong T^m$. We also arranged that this forcing is essentially a Souslin tree, and hence c.c.c. and countably distributive, and adds no new automorphisms of any T^i and no unwanted isomorphisms from T^i to T^j when $m \leq i < j$. So the theorem is proved. \square

Parts of this construction are redundant, and we don't actually need all the cases that we mentioned. For example, we needn't act explicitly to ensure that the T^n are rigid, since this will follow by our acting to ensure that they are $T^{0,\dots,m}$ -absolutely rigid. Also, the controller trees will automatically have the

unique branch property, since otherwise they would create unwanted isomorphisms or automorphisms. We included these redundant simpler cases in the construction simply because they may help to explain the later more complicated parts of the construction, and they certainly do no harm.

The construction suggests a degree of flexibility that might allow one to improve Theorem 5.3 by increasing ω to any countable ordinal or even ω_1 . For example, already the controller trees can be used to make any finitely many trees isomorphic, and one could arrange that this would not add unwanted isomorphisms or automorphisms. What is needed is a more complicated construction that would anticipate names for additional automorphisms after forcing with *infinite* products of controller trees. This would produce groups whose automorphism tower can be forced to have any countable ordinal as its height, thereby improving Theorem 5.2 from ω to ω_1 as well.

It appears that our methods may generalize to the case of higher cardinals, producing suitably rigid Souslin κ^+ -trees from a suitable \diamond_{κ^+} hypothesis, ultimately constructing groups, whose automorphism towers are highly malleable by forcing. We leave this idea for a subsequent project.

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